

Efficient Allocations in Dynamic Private Information Economies with Persistent Shocks: A First-Order Approach

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Abstract

I study efficient allocations in a dynamic private information economy with a continuum of individual shocks that are persistent. I formulate the problem recursively and develop a first-order approach to simplify it. The main advantage of the first-order approach is that it allows for a substantial reduction of the state space of the dynamic program. This makes the problem tractable and permits quantitative implementation of the problem. I find sufficient conditions that can be numerically computed to verify the validity of the first-order approach. To illustrate the benefits of the recursive formulation I numerically compute the efficient allocations in a taste shock economy with exponential utility and Pareto distributed shocks. I show that as the persistency of shocks increases, the insurance against taste shocks decreases. On the other hand, the efficient allocation tends to exhibit more insurance over time.

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1 Introduction

This paper studies efficient allocations in a dynamic economy with continuum of idiosyncratic private information shocks that follow a Markov process. It is well known that persistent shocks significantly complicate the analysis of the problem. As shown by [Fernandes and Phelan \(2000\)](#), the efficient allocations do have a recursive structure, but the dimensionality of the state space is proportional to the number of possible shock values. Quantitative analysis thus is feasible only for a small number of shocks. This paper shows that when the first order approach is used, the state space can be reduced to a manageable dimension of two endogenous state variables, even when the shocks take a continuum of values. This greatly increases the tractability of the problem and the ability to investigate the efficient allocations quantitatively.

It is well known from [Green \(1987\)](#), [Thomas and Worrall \(1990\)](#) or [Atkeson and Lucas \(1992\)](#) that when the shocks are i.i.d., the recursive formulation takes a very simple form. The agent's reporting history up to any period t can be summarized by a single statistic, promised utility, which is the lifetime utility the agent is entitled to receive from period t onwards. Based on the agent's report, the social planner chooses current consumption, current output and continuation utility from tomorrow on, such that the expected utility of the truthtelling agent equals her promised utility and it is in the agent's interest to report her shock truthfully. The continuation utility chosen by the social planner becomes promised utility at the beginning of the next period, and so on.

A key feature that allows for this recursive formulation is that preferences over the allocations from the current period onwards are identical for all agents, regardless of whether they have reported truthfully in previous periods or not. This feature no longer holds when shocks are persistent. The probability distribution and hence preferences over the allocations now depend on the last period shock. Thus, the continuation utility must depend both on the current report, and on the current

shock. Since the social planner does not observe the shock, incentive compatibility requires that he chooses a continuation utility for all possible last period shocks. With continuum of shocks, the state variable must therefore be a *function* of the last period shock. This function, called the promised utility function, specifies the agent's promised utility for each possible shock the agent might have had last period.

The advantage of the first-order approach in static economies is that it restricts attention to local deviations from the optimum (Mirrlees (1971)). This paper shows that there is an additional benefit from using the first-order approach in dynamic economies with persistent shocks: only the continuation utility of the truthtelling agent and its marginal change matter in determining whether an allocation satisfies the envelope condition. As a result, one only needs to keep track of the promised utility of the truthtelling agent and a marginal change in her promised utility, rather than keeping track of the promised utility function. Thus, the state space of the dynamic program is reduced from a function to two numbers.

To apply the first order approach, I first show that the continuation utility function is differentiable in the agent's shock whenever the transition function is differentiable in the previous shock. Assuming this holds, the first order approach is correctly defined. I then find conditions that guarantee that the first order approach is valid, and can be verified numerically.

I apply the first-order approach to study a simple economy with privately observed taste shocks that are serially correlated and have Pareto distributed innovations. I completely characterize the set of utilities that are implementable by the first-order approach by showing that its upper and lower boundaries are piecewise linear, and provide a simple algorithm for their computation. Knowing the set of implementable utilities is a necessary prerequisite for finding the efficient allocation, but it is also useful for analyzing how model parameters affect the set of social planner's choices. In particular, I show that the set of implementable utilities becomes smaller when the persistency of shocks decreases, and when the bounds on period utility tighten.

In the extreme case of permanent shocks, only autarchic allocations become feasible and no insurance is provided.

The efficient allocations are quantitatively analyzed for the case of exponential utility. I characterize the efficient allocations along two dimensions: by the amount of intratemporal insurance, measured by the correlation between consumption and the shocks, and by the amount of intertemporal insurance, measured by changes in the dispersion of the continuation utility. I find that the intratemporal insurance decreases with persistency of shocks. Higher persistency of shocks also implies less intertemporal insurance, because the continuation utility disperses faster when persistency is higher. On the other hand, the dynamics of the efficient allocation is such that, over time, more intratemporal insurance is provided.

The first-order approach has been so far used mainly in static environments. Dynamics extensions include [Courty and Li \(2000\)](#), who study a two period price discrimination problem, and especially [Pavan, Segal, and Toikka \(2008\)](#), who characterize both the necessary and sufficient conditions for the first-order approach in a much more general dynamic environment than the one here. Contrary to this paper, however, they do not characterize the recursive structure of the efficient allocations and of the recursive social planner's problem (since in general it will not exist).¹

The literature on recursive dynamic contracting with persistent private information has mainly focused on environment with finite number of shocks, starting with [Fernandes and Phelan \(2000\)](#). [Battaglini and Coate \(2008\)](#) study efficient income taxation in a Mirrleesian economy with hidden productivity, while [Tchisty \(2006\)](#) studies the optimal security design with hidden cash flows. Both papers exploit a very special environment where Markov shocks can take only two values and the agents are risk neutral. There is also a growing literature that studies private information

¹The first-order approach has also been recently used in related dynamic environments. [Werning \(2001\)](#) and [Abraham and Pavoni \(2006\)](#) use it to study dynamic hidden savings problem while [Jarque \(2002\)](#) uses it to analyze moral hazard environment with effort persistence.

economies with persistent shocks in a continuous time. [Zhang \(2007\)](#) characterizes the efficient allocations in a Mirrleesian economy with two shocks. [Williams \(2008\)](#) studies an economy with hidden income that follows a Brownian motion. As he points out, the trade-offs in continuous time models are very different from discrete time models. Despite that, some of his findings are similar to the findings of this paper, namely that less insurance is provided with higher shock persistency.

The paper is organized as follows. The model is introduced in the next section. Section 3 defines a recursive formulation of the constrained Pareto optimal allocations, and of the social planner's problem. Section 4 analyzes conditions that guarantee that the first-order approach is correctly defined, and presents necessary and sufficient conditions for the first-order approach to be valid. Section 5 then introduces a relaxed recursive formulation which builds on the first order approach, and shows that, as long as the first order approach is valid, the relaxed recursive formulation is equivalent to the recursive formulation of Section 3. The taste shock economy is analyzed in Section 6, and numerical results are presented in Section 7. Section 8 concludes. All proofs are relegated to the Appendix.

2 The Model

The world begins at time $t = 1$. The economy is populated by a continuum of infinitely lived agents of measure one. Their preferences in each period are defined over consumption $c \in \mathbb{R}_+$, output $y \in \mathbb{R}_+$, and depend on the productivity shock $\theta \in \Theta$. The period utility function is given by $U : \mathbb{R}_+ \times \mathcal{Y} \rightarrow \mathbb{R}$ where $\mathcal{Y} \subseteq \mathbb{R}_+ \times \Theta$ is the set of all feasible output-productivity shock pairs.² The utility function is increasing, concave and differentiable in consumption, decreasing, convex and differentiable in

²This formulation allows for the possibility that not all output levels are feasible for a given productivity shock. For example, in a standard Mirrleesian economy where labor supply $l = \frac{y}{\theta}$ is between 0 and 1, \mathcal{Y} is defined by the inequality $0 \leq y \leq \theta$.

output, and increasing and linear in the productivity shock.³ In addition, $\underline{u}_0 + \underline{u}\theta \leq U \leq \bar{u}_0 + \bar{u}\theta$, where \underline{u}_0 , \bar{u}_0 , \underline{u} and \bar{u} are finite constants. Agents discount future by a discount factor $\beta \in (0, 1)$. The set of lifetime utilities the agent may receive is given by \mathcal{W} .

The agents are heterogeneous with respect to their productivity shock $\theta_t \in \Theta = [\underline{\theta}, \bar{\theta}] \subseteq \mathbb{R}_+$. The productivity shock is a continuous random variable that follows a first-order Markov process with transition function $Q : \Theta \times \mathcal{B}_\Theta \rightarrow [0, 1]$, where \mathcal{B}_Θ is a Borel σ -algebra on Θ . It is assumed that Q is differentiable with respect to its first argument, and that $|Q_1|$ is bounded by an integrable function. For each $t \geq 1$, one can construct a probability measures μ^t over the shock sequences $\theta^t \in \Theta^t$ that follow an initial given shock θ_0 . Similarly, for $1 \leq j < t$ one can construct probability measures μ_j^t over the subsequences of shocks $\theta_j^t \in \Theta^{1+t-j}$, given the initial shock θ_0 .

The social planner is committed to deliver a lifetime utility $w_1 \in \mathcal{W}$. The objective of the social planner is to minimize the present value of his costs. He has access to a credit market and can freely borrow or save. The sequence of intertemporal prices of consumption is given by $q = \{q_t\}_{t=1}^\infty$, where $q_t \in (0, 1)$ is a relative price of consumption between periods t and $t + 1$. In what follows, the sequence of intertemporal prices will be taken as given. The results can be easily generalized for endogenous intertemporal prices.

The timing and information structure are as follows. At the beginning of each period, agents observe their current productivity shock. After that, a report is sent to the social planner, and consumption and output are determined. While both consumption and output are observed by the social planner, the productivity of the agent is her private information. The only exception is the initial shock θ_0 , which is observed by the social planner and is the same for everyone.

³Linearity in productivity shock is assumed for technical reasons. Two most common examples that satisfy this assumption is a taste shock specification without production, $U = \theta u(c)$, or with additively separable production, $U = u(c) - \theta v(y)$. Linearity in productivity shock can be relaxed if one allows for lotteries over consumption and output.

3 A Recursive Formulation

This section provides a recursive formulation of the constrained Pareto optimal allocations, and of the social planner's problem. The recursive formulation will not be derived from a primitive sequence formulation of the problem because it is a fairly straightforward generalization of the arguments laid out in [Atkeson and Lucas \(1992\)](#) and [Fernandes and Phelan \(2000\)](#), and is not central to the main point of this paper.⁴

In each period $t > 1$, the social planner's choice is conditioned on a function $\mathbf{w}(\cdot) : \Theta \rightarrow \mathcal{W}$, called a *promised utility function*. This function determines the agent's lifetime utility entitlement for each possible shock the agent may have received in the previous period: if last period's shock was $\hat{\theta}_-$, then the agent is entitled to receive lifetime utility $\mathbf{w}(\hat{\theta}_-)$. The promised utility function allows the social planner to deliver the promised utility to an agent for any last period shock, regardless of whether or not the agent has reported her shock truthfully.

Define an *allocation rule* $Z = \{C_t, Y_t, W_{t+1}\}_{t=1}^{\infty}$ as follows. Let \mathcal{L} be a space of functions $\mathbf{w} : \Theta \rightarrow \mathcal{W}$. For each $t > 1$, define a consumption function $C_t : \mathcal{L} \times \Theta^2 \rightarrow \mathbb{R}_+$, an output function $Y_t : \mathcal{L} \times \Theta^2 \rightarrow \mathbb{R}_+$, and a continuation utility function $W_{t+1} : \mathcal{L} \times \Theta^3 \rightarrow \mathcal{W}$. The continuation utility function $W_{t+1}(\mathbf{w}(\cdot), \theta_-, \hat{\theta}, \theta)$ determines the continuation utility of an agent who is promised $\mathbf{w}(\cdot)$, reported θ_- in the previous period, reports $\hat{\theta}$ in the current period, and her current shock is θ . The consumption and output functions $C_t(\mathbf{w}(\cdot), \theta_-, \hat{\theta})$ and $Y_t(\mathbf{w}(\cdot), \theta_-, \hat{\theta})$ determine the consumption and output of that agent. Unlike the continuation utility function, they do not depend on the current shock θ . For $t = 1$, the definition is modified to take into account that, at the beginning of period 1, only the promised utility $w_1 \in \mathcal{W}$ must be delivered: The first period allocation rule is given by $C_1 : \mathcal{W} \times \Theta^2 \rightarrow \mathbb{R}_+$, $Y_1 : \mathcal{W} \times \Theta^2 \rightarrow \mathbb{R}_+$ and $W_2 : \mathcal{W} \times \Theta^3 \rightarrow \mathcal{W}$.

⁴The relationship between the sequence formulation and the recursive formulation of this section can be found in [Kapička \(2008\)](#).

An allocation rule must be such that, for all $t > 1$, the expected utility of an agent with last period shock $\hat{\theta}_- \in \Theta$ is given by $\mathbf{w}(\hat{\theta}_-)$:⁵

$$\mathbf{w}(\cdot) = \int_{\Theta} [U(C_t(\theta), Y_t(\theta), \theta) + \beta W_{t+1}(\theta, \theta)] Q(\cdot, d\theta). \quad (1)$$

This is the promise keeping constraint in the usual sense only when $\mathbf{w}(\cdot)$ is evaluated at the previous period shock θ_- . For any other values $\hat{\theta}_- \neq \theta_-$ it corresponds to the threat keeping constraint in the sense of [Fernandes and Phelan \(2000\)](#), and determines what lifetime utility the agent would have been entitled to, if her previous period shock was $\hat{\theta}_-$, but she still reported θ_- . The first period needs to be treated differently since the time zero shock θ_0 is publicly known. The promise keeping constraint only requires that the agent must receive the promised utility w_1 :

$$w_1 = \int_{\Theta} [U(C_1(\theta), Y_1(\theta), \theta) + \beta W_2(\theta, \theta)] Q(\theta_0, d\theta). \quad (2)$$

Since the productivity shock is a private information of the agent, the allocation must be such that the agent always prefers to report her shock truthfully. The temporary incentive compatibility constraint requires that for all $t \geq 1$,

$$U(C_t(\theta), Y_t(\theta), \theta) + \beta W_{t+1}(\theta, \theta) \geq U(C_t(\hat{\theta}), Y_t(\hat{\theta}), \theta) + \beta W_{t+1}(\hat{\theta}, \theta) \quad \forall \hat{\theta}, \theta \in \Theta^2. \quad (3)$$

To rule out the possibility that the social planner chooses a continuation utility function that will later be impossible to deliver, one needs to impose an additional restriction on the social planner's set of choices: For all $t \geq 1$, it is required that

$$W_{t+1}(\theta, \cdot) \in \mathcal{U}^* \quad \forall \theta \in \Theta, \quad (4)$$

where the set of implementable allocations \mathcal{U}^* is a fixed point of the following operator:

$$T\mathcal{U} = \{\mathbf{w} \in \mathcal{L}: \exists C: \Theta \rightarrow \mathbb{R}_+, Y: \Theta \rightarrow \mathbb{R}_+, W': \Theta^2 \rightarrow \mathbb{R} \quad (5)$$

$$\text{such that (1) and (3) holds and } W'(\theta, \cdot) \in \mathcal{U} \forall \theta \in \Theta\}. \quad (6)$$

Standard arguments of [Abreu, Pearce, and Stacchetti \(1990\)](#) imply that the set \mathcal{U}^* is nonempty, convex and compact.

⁵For notational convenience, the state variables $\mathbf{w}(\cdot)$ and θ_- are kept implicit whenever possible.

3.1 The Social Planner's Problem

The Bellman principle of optimality implies that the social planner's cost minimization problem can be written recursively. For $t \geq 1$ define a cost function $V_t : \mathcal{U}^* \rightarrow \mathbb{R}$ to be the minimal costs of delivering a promised utility function $\mathbf{w}(\cdot)$ to an agent who has reported a shock θ_- in period $t - 1$:

$$V_t(\mathbf{w}(\cdot), \theta_-) = \min_{C, Y, W'} \int_{\Theta} [C(\theta) - Y(\theta) + q_t V_{t+1}(W'(\theta, \cdot), \theta)] Q(\theta_-, d\theta), \quad (7)$$

subject to the temporary incentive compatibility constraint (3), the promise keeping constraint (1) and the implementability constraint (4). In the first period, the cost minimization problem of the social planner is given by

$$V(w_1, \theta_0) = \min_{\mathbf{w}(\cdot) \in \mathcal{U}^*} V_1(\mathbf{w}(\cdot), \theta_0) \quad s.t. \quad \mathbf{w}(\theta_0) = w_1. \quad (8)$$

4 First-Order Approach

While the recursive approach of the previous section works well in theory, it is clear that having a function as a state variable prevents one not only from solving for the efficient allocations numerically, but also from gaining more insights into the efficient mechanism. To deal with this weakness, I now develop a first-order approach to simplify the problem. The advantage of the first-order approach is that, as long as it is justified, one does not need to specify the continuation utility function W_{t+1} for the off-equilibrium values. This in turn implies that one does not need to specify the off-equilibrium values of the state variable as well. Rather, the recursive structure of the problem will depend only on the equilibrium promised utility $\mathbf{w}(\theta_-)$ of the truth-telling agent and on her marginal promised utility $\frac{\partial}{\partial \theta} \mathbf{w}(\theta_-)$, instead of on the whole promised utility function $\mathbf{w}(\cdot)$. A corresponding recursive cost minimization problem will then depend only on a handful of simple state variables and can be numerically computed and analyzed.

The application of the first-order approach is complicated by the fact that, as follows from (3), the lifetime utility of the agent is a sum of two terms: $U(C_t(\hat{\theta}), Y_t(\hat{\theta}), \theta)$ and $W_{t+1}(\hat{\theta}, \theta)$. The first term is just a period utility function and its properties of interest, namely differentiability with respect to θ , can be imposed exogenously. But $W_{t+1}(\hat{\theta}, \theta)$ is endogenous to the social planner's problem and its properties are not immediately known. Fortunately, as shown in the next Lemma, the set \mathcal{U}^* contains only functions that are differentiable in θ .

Lemma 1. *Suppose that $W'(\hat{\theta}, \cdot) \in \mathcal{U}^*$ for all $\hat{\theta} \in \Theta$. Then $W'(\hat{\theta}, \theta)$ is differentiable with respect to θ for all $\hat{\theta} \in \Theta$. In addition, $\frac{\partial}{\partial \theta} W'(\hat{\theta}, \theta)$ is bounded by an integrable function.*

The intuition for the first part of the Lemma is that, for a given report $\hat{\theta}$, the function $W_{t+1}(\hat{\theta}, \cdot)$ needs to satisfy (1) at the beginning of period $t + 1$. Hence, if the right-hand side of (1) is differentiable in θ , the left-hand side must be as well. But the right-hand side of (1) depends on θ only through the transition function Q . Hence, if the function $Q(\theta, \cdot)$ is differentiable in θ , which it is by assumption, then $W_{t+1}(\hat{\theta}, \theta)$ is also differentiable in θ .

The next theorem builds on Lemma (1) and derives necessary conditions for an allocation rule to be incentive compatible. The proof is similar in many aspects to the one used in static economies, e.g. in [Mirrlees \(1986\)](#).

Theorem 2. *If an allocation rule \tilde{z} satisfies the temporary incentive compatibility constraint (3) for $t \geq 1$ then*

$$U(C_t(\theta), Y_t(\theta), \theta) + \beta W_{t+1}(\theta, \theta) = \int_{\underline{\theta}}^{\theta} [U_{\theta}(C_t(\varepsilon), Y_t(\varepsilon)) + \beta \frac{\partial}{\partial \theta} W_{t+1}(\varepsilon, \varepsilon)] d\varepsilon + W_0. \quad (9)$$

The envelope condition (9) is obtained by applying the Envelope Theorem to the incentive compatibility constraint (3).⁶ It shows how lifetime utility must vary with the current period shock in order to be incentive compatible. The expression under

⁶ Note that since U is linear in θ , U_{θ} depends only on C and Y .

the integral consists of two terms. The first term is the current informational rent: the extra current period utility an agent receives from a marginal increase in current skills. The second term corresponds to the future informational rent: extra lifetime utility from tomorrow on that an agent receives from a marginal shift in the future distribution of shocks. The second term is zero if shocks are i.i.d. because current private information bears no information about future shocks.

The envelope condition is clearly necessary, but not sufficient. A sufficient condition for an allocation to be incentive compatible is given next:

Theorem 3. *If an allocation rule satisfies the envelope condition (9) and, in addition,*

$$U_{\theta}(C_t(\hat{\theta}), Y_t(\hat{\theta})) + \beta \frac{\partial}{\partial \theta} W_{t+1}(\hat{\theta}, \theta) \quad (10)$$

is increasing in $\hat{\theta}$ for all $\theta \in \Theta$, then the allocation rule satisfies the temporary incentive compatibility constraint (3).

If the shocks are i.i.d., the second term drops out and condition (10) is equivalent to a simple requirement that $Y_t(\theta)$ is increasing in θ , and one can also show that this requirement is necessary. When the shocks are persistent, monotonicity of Y_t is neither required nor implied. In principle, one can have an incentive compatible allocation rule such that Y_t is decreasing, if $\frac{\partial}{\partial \theta} W_{t+1}$ increases sufficiently fast.

5 A Relaxed Recursive Formulation

In what follows, I will consider a *relaxed social planner's problem* where the monotonicity condition (10) is ignored. Validity of the relaxed social planner's problem will only be checked after the solution is found.

The envelope condition (9) suggests that when the incentive compatibility constraint (3) is replaced by the envelope condition (9), everything except for the continuation utility and the marginal continuation utility of the truthtelling agent is irrelevant for the recursive formulation of the problem. With that in mind, define

a *relaxed allocation rule* $z = \{c_t, y_t, w_{t+1}, g_{t+1}\}_{t=1}^{\infty}$ as follows. Let $\mathcal{G} \subseteq \mathbb{R}$ be the set of all feasible marginal promised utilities. For each $t > 1$, the relaxed allocation rule consists of the functions $c_t : \mathcal{W} \times \mathcal{G} \times \Theta^2 \rightarrow \mathbb{R}_+$, $y_t : \mathcal{W} \times \mathcal{G} \times \Theta^2 \rightarrow \mathbb{R}_+$, $w_{t+1} : \mathcal{W} \times \mathcal{G} \times \Theta^2 \rightarrow \mathcal{W}$ and $g_{t+1} : \mathcal{W} \times \mathcal{G} \times \Theta^2 \rightarrow \mathcal{G}$. For $t = 1$ the relaxed allocation rule is defined by $c_1 : \mathcal{W} \times \Theta^2 \rightarrow \mathbb{R}_+$, $y_1 : \mathcal{W} \times \Theta^2 \rightarrow \mathbb{R}_+$, $w_{t+1} : \mathcal{W} \times \Theta^2 \rightarrow \mathcal{W}$ and $g_{t+1} : \mathcal{W} \times \Theta^2 \rightarrow \mathcal{G}$. The function $w_{t+1}(w, g, \theta_-, \theta)$ now represents the continuation utility of the truthtelling agent, while $g_{t+1}(w, g, \theta_-, \theta)$ expresses how the continuation utility of the truthtelling agent changes with her shock. Compared to the allocation rule, the domain of the relaxed allocation rule has been simplified for $t > 1$ because the promised utility function $\mathbf{w}(\cdot)$ has been replaced by the promised utility w and the marginal promised utility g .

The relaxed allocation rule z is required to satisfy four conditions. The first is the envelope condition: For all $t \geq 1$,

$$U(c_t(\theta), y_t(\theta), \theta) + \beta w_{t+1}(\theta) = \int_{\underline{\theta}}^{\theta} [U_{\theta}(c_t(\varepsilon), y_t(\varepsilon)) + \beta g_{t+1}(\varepsilon)] d\varepsilon + w_0, \quad (11)$$

which is equation (9) applied to the relaxed allocation rule. The second requirement is that for all $t \geq 1$, the promise keeping constraint must hold:

$$w = \int_{\Theta} [U(c_t(\theta), y_t(\theta), \theta) + \beta w_{t+1}(\theta)] Q(\theta_-, d\theta). \quad (12)$$

Note that, unlike constraint (1), the promise keeping constraint is required to hold only for the last period shock of the truthtelling agent θ_- .

The third constraint, called the *marginal promise keeping constraint*, is new. For all $t > 1$, the social planner is restricted to increase the marginal continuation value of an agent with last period shock θ_- at rate g :

$$g = \int_{\Theta} [U(c_t(\theta), y_t(\theta), \theta) + \beta w_{t+1}(\theta)] Q_1(\theta_-, d\theta). \quad (13)$$

Finally, the last requirement is that for all $t > 1$ the relaxed allocation rule must be such that next period promised utility $w_{t+1}(\theta)$ and marginal promised utility $g_{t+1}(\theta)$

can be delivered. The implementability constraint is

$$[w_{t+1}(\theta), g_{t+1}(\theta)] \in \mathcal{V}^*(\theta) \quad \forall \theta \in \Theta, \quad (14)$$

where the set of implementable utilities $\mathcal{V}^*(\theta) \subseteq \mathcal{W} \times \mathcal{G}$ is the fixed point of the following operator:

$$\begin{aligned} T\mathcal{V}(\theta_-) = \{ & (w, g) \in \mathcal{W} \times \mathcal{G} : \exists c : \Theta \rightarrow \mathbb{R}_+, y : \Theta \rightarrow \mathcal{R}_+, w' : \Theta \rightarrow \mathcal{W}, g' : \Theta \rightarrow \mathcal{G} \\ & \text{such that (12), (13) and (11) holds and } [w'(\theta), g'(\theta)] \in \mathcal{V}(\theta)\}. \end{aligned} \quad (15)$$

Before the relaxed social planner's problem can be solved, one needs to characterize the set of implementable utilities \mathcal{V}^* .

5.1 Characterizing \mathcal{V}^*

The set \mathcal{V}^* can be shown to be nonempty, compact and convex for all $\theta \in \Theta$. Bounds on \mathcal{V}^* can be found as follows. If the envelope condition (11) and the marginal promise keeping constraint (13) are both ignored, the lowest possible lifetime utility that can be delivered by the social planner is given by a function $\underline{w}(\theta_-)$ that satisfies

$$\underline{w}(\theta_-) = \int_{\Theta} [U(\underline{c}, \bar{y}, \theta) + \beta \underline{w}(\theta)] Q(\theta_-, d\theta). \quad (16)$$

The highest possible lifetime utility $\bar{w}(\theta_-)$ is defined similarly. To compute bounds on the marginal promised utility, use the envelope condition (11) to replace the term under the integral on the right-hand side of (13) and integrate by parts to get the following alternative expression for the marginal promise keeping constraint:

$$g = \int_{\Theta} [U_{\theta}(c(\theta), y(\theta)) + \beta g'(\theta)] \bar{Q}_1(\theta_-, d\theta), \quad (17)$$

where $\bar{Q}(\theta_-, \theta) = 1 - Q(\theta_-, [\theta, \theta])$, and so $\bar{Q}_1(\theta_-, \theta) = -Q_1(\theta_-, [\theta, \theta])$. The marginal promised utility must then be greater than the minimum of (17), given by

$$\underline{g}(\theta_-) = \int_{\Theta} [U_{\theta}(\underline{c}, \bar{y}) + \beta \underline{g}(\theta)] \bar{Q}_1(\theta_-, d\theta).$$

The upper bound on the marginal promised utility $\bar{g}(\theta)$ is defined analogously.

The set $[\underline{w}(\theta), \bar{w}(\theta)] \times [\underline{g}(\theta), \bar{g}(\theta)]$ then defines a "bounding box" on the set of implementable utilities $\mathcal{V}^*(\theta)$. One can then start with the bounding box and iterate on the operator T defined by (15) to obtain \mathcal{V}^* . That may, however, be computationally intensive. A sharper characterization of \mathcal{V}^* can be obtained by analyzing its boundaries directly. A lower boundary $\underline{\mathcal{V}}(g, \theta) : [\underline{g}(\theta), \bar{g}(\theta)] \times \Theta \rightarrow [\underline{w}(\theta), \bar{w}(\theta)]$ is the smallest possible promised utility that is implementable when the marginal promised utility is g . Since a continuation of an allocation on the lower boundary is itself on the lower boundary, $\underline{\mathcal{V}}(g, \theta)$ satisfies the following Bellman equation:

$$\underline{\mathcal{V}}(g, \theta_-) = \min_{c, y, g'} \int_{\Theta} [U(c(\theta), y(\theta), \theta) + \beta \underline{\mathcal{V}}(g'(\theta), \theta)] Q(\theta_-, d\theta) \quad (18)$$

subject to $g'(\theta) \in [\underline{g}(\theta), \bar{g}(\theta)]$, the marginal promise keeping constraint (17), and a requirement that the allocation on the lower boundary satisfies the envelope condition:

$$U(c(\theta), y(\theta), \theta) + \beta \underline{\mathcal{V}}(g'(\theta), \theta) = \int_{\underline{\theta}}^{\theta} [U_{\theta}(c(\varepsilon), y(\varepsilon)) + \beta g'(\varepsilon)] d\varepsilon + \underline{w}(\theta_-). \quad (19)$$

The upper boundary $\bar{\mathcal{V}}(g, \theta)$ is defined similarly.

Since \mathcal{V}^* is a convex set, the upper boundary must be concave, and the lower boundary must be convex. It can also be shown that the lower left-hand corner $(\underline{w}(\theta_-), \underline{g}(\theta_-))$ and the upper right-hand corner $(\bar{w}(\theta_-), \bar{g}(\theta_-))$ both belong to $\mathcal{V}^*(\theta)$ ⁷. Although the contracts that lie on those boundaries will in general not be simple (e.g. report independent), computation of those boundaries is clearly a relatively simple problem. Moreover, in special cases one can obtain much sharper characterization of those boundaries. As we will see in the next section, a very simple characterization of \mathcal{V}^* is available if the shocks are Pareto distributed.

⁷This follows from the fact that $(\underline{w}(\theta_-), \underline{g}(\theta_-))$ can be attained by report independent consumption and output pair (\underline{c}, \bar{y}) , and so is trivially incentive compatible. Similar argument establishes that $(\bar{w}(\theta_-), \bar{g}(\theta_-))$ is incentive compatible.

5.2 The Relaxed Social Planner's Problem

The social planner's cost function $v_t : \mathcal{W} \times \mathcal{G} \times \Theta \rightarrow \mathbb{R}$ is given, for $t > 1$, recursively by

$$v_t(w, g, \theta_-) = \min_{c, y, w', g'} \int_{\Theta} [c(\theta) - y(\theta) + q_t v_{t+1}(w'(\theta), g'(\theta), \theta)] Q(\theta_-, d\theta), \quad (20)$$

subject to (12), (13), (11) and (14). The first period cost function $v : \mathcal{W} \times \Theta$ does not depend on the marginal promised utility and is given by

$$v(w_1, \theta_0) = \min_{g \in \mathcal{G}} v_1(w_1, g, \theta_0) \quad s.t. \quad (w_1, g) \in \mathcal{V}^*(\theta_0). \quad (21)$$

To assess the validity of the relaxed social planner's problem, one needs to characterize the lifetime utility from all possible deviations. Consider an agent who reports $\hat{\theta}$ in the current period, although her true shock is θ . Under the relaxed social planner's problem, her next period states are given by $w_{t+1}(\hat{\theta})$, $g_{t+1}(\hat{\theta})$ and $\hat{\theta}$. Consequently, the deviator's continuation utility $\hat{w}_{t+1}(\theta, \hat{\theta})$ is

$$\hat{w}_{t+1}(\theta, \hat{\theta}) = \int_{\Theta} \{U[c_t(w_{t+1}(\hat{\theta}), g_{t+1}(\hat{\theta}), \hat{\theta}, \theta'), y_t(w_{t+1}(\hat{\theta}), g_{t+1}(\hat{\theta}), \hat{\theta}, \theta'), \theta'] \quad (22)$$

$$+ \beta w_{t+1}(w_{t+1}(\hat{\theta}), g_{t+1}(\hat{\theta}), \hat{\theta}, \theta')\} Q(\theta, d\theta'). \quad (23)$$

A first-order approach will be called *valid* if its solution satisfies a condition analogous to the monotonicity condition (10), but with the continuation utility given by function \hat{w}_{t+1} :

Definition 4. The first-order approach is valid if

$$U_{\theta}(c_t(\hat{\theta}), y_t(\hat{\theta})) + \beta \frac{\partial}{\partial \theta} \hat{w}_{t+1}(\theta, \hat{\theta}) \quad (24)$$

increases in $\hat{\theta}$ for all $\theta \in \Theta$.

One may guess that the set of relaxed allocation rules is in some sense identical to the set of allocation rules or allocations. That is, however, not true. As long as the first-order approach is valid, one can choose any relaxed allocation rule and

construct an allocation rule that delivers the same consumption and output assignment after each history of reports, but the reverse is not true. The reason is that the first-order approach is unable to provide arbitrary restrictions off the equilibrium path. However, as shown in the next theorem, these off-equilibrium restrictions are irrelevant: they can only increase the costs to the social planner and will therefore never be used in the optimum. Hence the recursive formulation relying on the first-order approach is sufficient to find the efficient allocation.

Theorem 5. *Suppose that the relaxed social planner's problem is valid. Then $v = V$.*

The intuition behind the proof is the following. For the efficient allocation rule one can define a relaxed allocation rule such that it satisfies (11), (12), (13) and (14), and delivers the same lifetime utility with the same costs. Since the relaxed allocation rule may not be efficient, $V \geq v$. Conversely, for the efficient relaxed allocation rule, one can find an allocation rule that satisfies (1) and, because the first-order approach is valid, (3) and (4). Thus, $v \geq V$.

6 A Taste Shock Economy

Consider a simple economy with multiplicative shocks, labeled as taste shocks. With some abuse of notation write $U(c, y, \theta) = \theta U(c)$. Assume that the social planner chooses the current period utility $u = U(c) \in [\underline{u}, \bar{u}]$ directly,⁸ and consumption is determined indirectly by $c = U^{-1}(u)$. The taste shocks are Pareto distributed with the density function given by

$$f(\theta_-, \theta) = \frac{\eta}{\theta} \left(\frac{\theta}{k\theta_-^\rho} \right)^{-\eta} \quad \theta \geq k\theta_-^\rho,$$

where $\eta > 1$ is the shape parameter. I set $k = \frac{\eta-1}{\eta} > 0$, which implies that $E(\theta|\theta_-) = \theta_-^\rho$, and so ρ is persistence of the shocks.

⁸It is customary to call u the utility, although the period utility is in fact θu .

The promise keeping constraint (12) can now be written as

$$w = \int_{k\theta_-}^{\infty} [\theta u(\theta) + \beta w'(\theta)] f(\theta_-, \theta) d\theta. \quad (25)$$

Specializing the marginal threat keeping constraint (13) for Pareto distributed shocks yields the following expression:

$$g = \rho \frac{\eta}{\theta_-} (w - w_0). \quad (26)$$

The marginal promised utility is thus proportional to the spread between the expected utility and the utility of the agent with the lowest possible shock, w_0 . The social planner's commitment to deliver a marginal promised utility g can, for a given w , alternatively be seen as a commitment to deliver a lower bound on agent utilities w_0 . The equation 26 also indicates that higher marginal continuation utilities will be associated with lower spread in the promised utilities.

6.1 Revisiting \mathcal{V}^*

The marginal promise keeping constraint (26) can be alternatively written by specializing the expression (17) and using the fact that, with Pareto distribution, $\bar{Q}_1(\theta_-, \theta) = \rho \frac{\theta}{\theta_-} f(\theta_-, \theta)$:

$$g = \frac{\rho}{\theta_-} \int_{k\theta_-}^{\infty} [\theta u(\theta) + \beta \theta g'(\theta)] f(\theta_-, \theta) d\theta. \quad (27)$$

Note that this expression for the marginal promise keeping constraint, as well as the right-hand side of the boundary value function (18) are both linear in $\theta u f$. Thus, if one ignores the envelope constraint (19), one can always choose a solution to (18) that is independent of the current report. But such a solution is trivially incentive compatible and so satisfies (19) as well. This makes the set of implementable utilities \mathcal{V}^* easy to characterize: Its lower and upper boundaries are continuous and piecewise linear functions. To show the result for the lower boundary, let $\underline{w}_0(\theta_-) = \underline{w}(\theta_-)$,

$g_0(\theta_-) = \underline{g}(\theta_-)$, and define two sequences $\{\underline{w}_i(\theta_-), \underline{g}_i(\theta_-)\}_{i=1}^\infty$ as follows:

$$\begin{aligned}\underline{w}_i(\theta_-) &= \bar{u}\theta_-^\rho + \beta \int_{k\theta_-^\rho}^\infty \underline{w}_{i-1}(\theta_-) f(\theta_-, \theta) d\theta \\ \underline{g}_i(\theta_-) &= \frac{\rho}{\theta_-} [\bar{u}\theta_-^\rho + \beta \int_{k\theta_-^\rho}^\infty \theta \underline{g}_{i-1}(\theta_-) f(\theta_-, \theta) d\theta].\end{aligned}$$

In words, $\underline{w}_0(\theta_-)$ represents the lifetime utility an agent with last period shock θ_- obtains from receiving the lowest period utility \underline{u} in both current and all future periods. $\underline{w}_1(\theta_-)$ represents the lifetime utility from receiving the highest period utility \bar{u} in the current period and reverting to the lowest period utility \underline{u} afterwards. Similarly, $\underline{w}_i(\theta_-)$ is the value of having \bar{u} for the first i periods and then reverting to \underline{u} . The sequence $\{\underline{g}_i(\theta_-)\}_{i=1}^\infty$ is a sequence of associated marginal promised utilities. Those sequences are important because they define the breakpoints of the lower boundary of the set $\mathcal{V}^*(\theta_-)$:

Lemma 6. $\underline{\mathcal{V}}(g, \theta_-)$ is piecewise linear, with breakpoints at $\{\underline{g}_i(\theta_-)\}_{i=0}^\infty$, and values at those breakpoints given by $\underline{\mathcal{V}}(\underline{g}_i(\theta_-), \theta_-) = \underline{w}_i(\theta_-)$.

A typical lower and upper boundary of \mathcal{V}^* is depicted in [Figure 1](#). It is easy to show that the slope of the first linear segment on the lower boundary is $\frac{\theta_-}{\rho}$, the slope of the second linear segment is $\frac{\theta_-}{\rho^2}$, and so on. The allocations that lie on the lower boundary are extremal allocations in the sense that either the current period utility or the marginal continuation utility is always at its highest or lowest possible value. If $g \in [\underline{g}_0(\theta_-), \underline{g}_1(\theta_-))$ then the allocation on the lower boundary consists of a period utility $u \in [\underline{u}, \bar{u})$, and a minimal marginal continuation utility $g'(\theta_-) = \underline{g}(\theta_-)$. If $g \in [\underline{g}_i(\theta_-), \underline{g}_{i+1}(\theta_-))$ for $i > 0$, the allocation is instead given by the maximal period utility $u = \bar{u}$, and the marginal continuation utility lies in the previous linear segment $g'(\theta) \in [\underline{g}_{i-1}(\theta_-), \underline{g}_i(\theta_-))$. The allocation on the lower boundary thus "drifts" downwards and in $i + 1$ periods converges to the lower left-hand corner of the set \mathcal{V}^* . Symmetrically, the upper boundary is a piecewise linear function with the slope of the i -th segment (starting in the upper right-hand corner) being $\frac{\theta_-}{\rho^i}$, the allocation

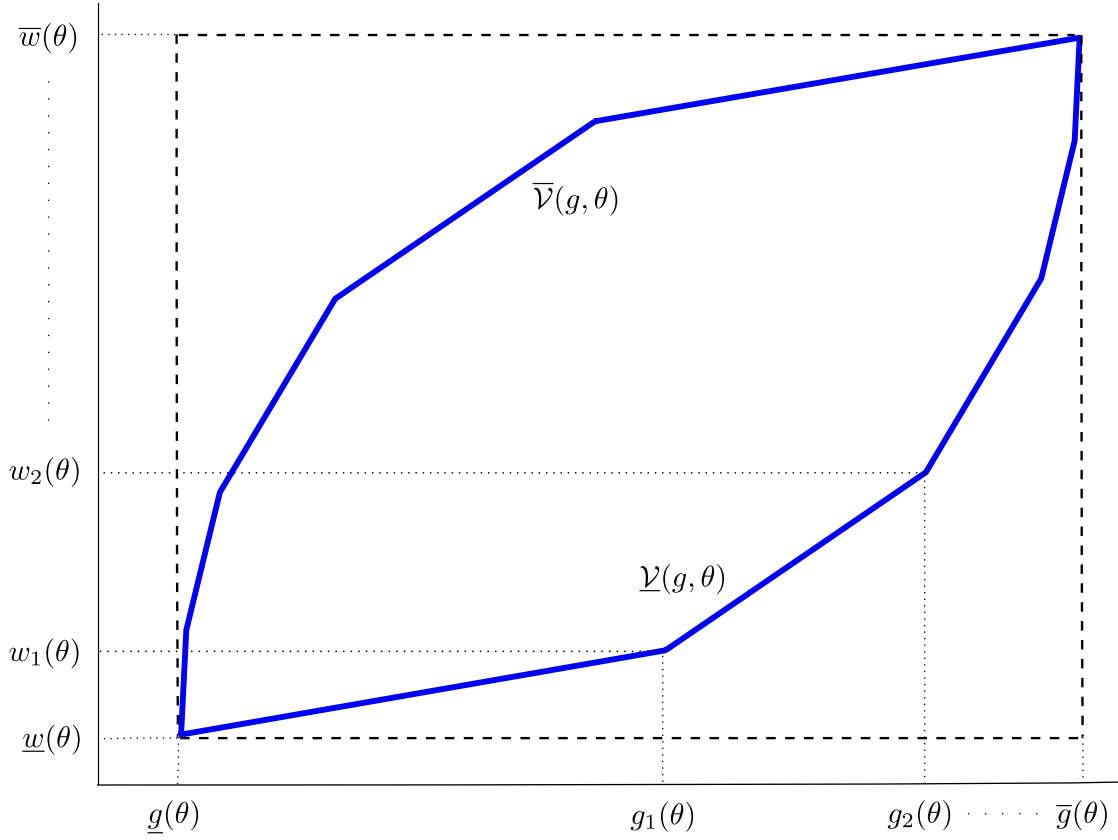


Figure 1: The Set of Implementable Utilities

on the upper boundary "drifts" upward and converges in $i + 1$ steps to the upper right-hand corner of \mathcal{V}^* .

How does the set of implementable utilities \mathcal{V}^* change if either the lower or upper bound on utility or the persistency of shocks change? Since the lower left-hand corner of \mathcal{V}^* is proportional to \underline{u} , a decrease in \underline{u} increases the set of implementable utilities. If \underline{u} is equal to minus infinity, then both the lower left-hand corner and lower boundary are equal to minus infinity as well. Since there is no lower bound on utility, the upper boundary now consists of only the first segment with the slope of $\frac{\theta_-}{\rho}$. The set of implementable utilities is then extremely easy to characterize: for $g < \bar{g}(\theta_-)$, any $w \leq \bar{g}(\theta_-) + \frac{\theta_-}{\rho}(g - \bar{g}(\theta_-))$ is implementable. The case of a change in the upper bound is symmetric.

When the shock persistency ρ approaches zero, the curvature of the boundaries increases because $\frac{\theta_-}{\rho^i}$ approaches infinity faster. Relative to the bounding box, the set of implementable utilities \mathcal{V}^* therefore increases. At the same time, since the marginal promised utility is proportional to ρ , the bounding box becomes horizontally more compressed. In the limiting case of serially uncorrelated shocks the bounding box collapses to a vertical line at $g = 0$ and any promised utility on that vertical line belongs to \mathcal{V}^* . When the shock persistency ρ approaches 1 then the curvature of the boundaries decreases. For $\rho = 1$ both boundaries coincide with the diagonal of the bounding box and collapse to each other. This implies that only allocations on the boundaries are feasible. But since all the allocations on the boundary consist of period utilities that are independent of the report, no insurance against the taste shocks is possible when $\rho = 1$. This implies the following remark:

Remark. If $\rho = 1$ then autarchy is the only implementable allocation.

Previous two paragraphs do not make clear what happens if \underline{u} equals to minus infinity and, at the same time, ρ equals one. Is the set of implementable utilities equal to the diagonal $\bar{g}(\theta_-) + \theta_-(g - \bar{g}(\theta_-))$ for $g < \bar{g}(\theta_-)$ as the case of $\rho = 1$ would suggest, or is it equal to all the points under this diagonal, as $\underline{u} = -\inf$ would suggest? The right answer is that the set of implementable utilities is only equal to the diagonal. Therefore, for $\underline{u} = -\inf$, there is a discontinuity in the set of implementable utilities \mathcal{V}^* as ρ approaches 1. The intuitive explanation for the discontinuity can be given as follows. As can be seen from (25) and (27), while the promised utility is discounted at rate β , the marginal promised utility is discounted at rate $\beta\rho$. Now consider an allocation such that the period utility grows at rate $\gamma \in [1, \frac{1}{\beta}]$ over time. The lifetime utility is then proportional to $\frac{1}{1-\beta\gamma}$. Since the marginal promised utility is discounted at rate $\beta\rho$, it is proportional to $\frac{1}{1-\beta\rho\gamma}$. The ratio of the promised utility to the marginal promised utility is therefore $\frac{1-\beta\rho\gamma}{1-\beta\gamma}$. When $\rho < 1$, a suitable choice of γ can make this ratio (effectively, the set of implementable utilities) to be anything between $\frac{1-\beta\rho}{1-\beta}$ and infinity. However, if $\rho = 1$, this ratio is always equal to 1 and the

set of implementable utilities discontinuously shrinks.

6.2 The Relaxed Social Planner's Problem

This section assumes that the period utility is exponential: $U(c) = -\frac{1}{\gamma}e^{-\gamma c}$. No upper or lower bound on consumption is assumed and so $\underline{u} = -\infty$ and $\bar{u} = 0$. The intertemporal price of consumption q_t is equal to the discount factor β in all periods, and so the efficient relaxed allocation rule, as well as the value function v are time independent.

The main benefit of the exponential utility is that one can get an almost closed form solution for the value function and for the policy functions. In particular, the value function is logarithmic in both the promised utility and the last period shock:

$$v(w, g, \theta_-) = -\frac{1}{1-\beta\gamma} \frac{1}{\gamma} \ln(-w) + \frac{\rho}{1-\beta\rho\gamma} \frac{1}{\gamma} \ln \theta_- + v(-1, \frac{g\theta_-}{-w}, 1),$$

and the policy functions have the following functional forms:

$$\begin{aligned} u(w, g, \theta_-, \theta) &= -\frac{w}{\theta_-^\rho} u(-1, \frac{g\theta_-}{-w}, 1, \frac{\theta}{\theta_-^\rho}), \\ w'(w, g, \theta_-, \theta) &= -ww'(-1, \frac{g\theta_-}{-w}, 1, \frac{\theta}{\theta_-^\rho}), \\ g'(w, g, \theta_-, \theta) &= -\frac{w}{\theta_-^\rho} g'(-1, \frac{g\theta_-}{-w}, 1, \frac{\theta}{\theta_-^\rho}). \end{aligned}$$

The optimal policy functions are thus homogeneous of degree one in w and g . In addition, the continuation utility is homogeneous of degree one in w, θ_- , and $\theta^{\frac{1}{\rho}}$, and the current utility and the marginal promised utility are homogeneous of degree $1 - \rho$ in w, θ_- , and $\theta^{\frac{1}{\rho}}$. Finally, the set \mathcal{V}^* is homogeneous as well in a sense that if $(w, g) \in \mathcal{V}^*(\theta_-)$ then $(-1, \frac{g\theta_-}{-w}) \in \mathcal{V}^*(1)$.

As follows from those properties, one can normalize the promised utility w to -1 , and the last period shock θ_- to 1, and to solve for a significantly reduced problem. Abusing the notation, drop those redundant state variables by writing $v(g) = v(-1, g, 1)$, and similarly for the policy functions. In addition, replace the

marginal continuation utility by a new variable h defined as $h(g, \theta) = \frac{g'(g, \theta)\theta}{-w'(g, \theta)}$. The variable h rescales the marginal continuation utility to the same units that the normalized problem is expressed in, and is therefore easily comparable to the current marginal promised utility. The value function v now satisfies the following Bellman equation:

$$v(g) = \min_{u, w', h} \int_k^\infty \left[-\frac{1}{\gamma} \ln(-\gamma u(\theta)) - \frac{\beta}{1 - \beta} \frac{1}{\gamma} \ln(-w'(\theta)) + \beta v(h(\theta)) \right] f(1, \theta) d\theta + K \quad (28)$$

where K is a constant, and the minimization is taken subject to three constraints: First, the solution must deliver the promised utility -1 ,

$$-1 = \int_k^\infty [\theta u(\theta) + \beta w'(\theta)] f(1, \theta) d\theta.$$

Second, the envelope condition and the marginal promise keeping constraint (26) must hold. By eliminating w_0 from both, one can combine them to the following single constraint:

$$\theta u(\theta) + \beta w'(\theta) = \int_k^\theta [u(\varepsilon)\varepsilon - \beta h(\varepsilon)w'(\varepsilon)] \frac{d\varepsilon}{\varepsilon} - 1 - \frac{g}{\eta\rho}.$$

Third, the continuation utility and the marginal continuation utility must satisfy the implementability constraint. Using the definition of h and the homogeneity property of the set \mathcal{V}^* , the implementability constraint can be written, for $\rho < 1$, as

$$h(\theta) \in [-\rho, 0).$$

There are several notable properties of the dynamic program (28). The value function v reaches its minimum at some $g^* \in [-\rho, 0)$. The first-order conditions imply that as θ goes to infinity, the marginal continuation utility h converges to zero. Moreover, if $g \geq g^*$ one can show that $h(\theta) \geq g^*$ for all $\theta \geq k$. This result is important because it implies that the upward sloping part of the value function $[g^*, 0)$ constitutes an ergodic set: once the marginal promised utility is in this set, it will never fall below g^* . Since in the first period the social planner chooses $g = g^*$,

the efficient allocation will always lie in the ergodic set. In addition, the marginal promised utility can never decrease in the future.

Increasing marginal continuation utility has one important implication: As follows from equation (26), high marginal continuation utilities are associated with low spread between the lifetime utility of the agent with the lowest possible shock, and the promised utility. This indicates that there is less dispersion in the continuation utilities. One should therefore expect that the efficient allocation will deliver more insurance against shocks over time.⁹

When the marginal promised utility increases, one can show that the current utility for the lowest possible shock $ku(k)$ increases relatively to the continuation utility $w'(k)$. This suggests that, for higher marginal promised utility the efficient allocation becomes more "front-loaded", with relatively more resources given to the agent in the current period. The reason behind this lies again in the fact that the marginal promised utility is discounted at higher rate than the promised utility.¹⁰

7 Numerical Simulations

This section provides a quantitative analysis of the efficient allocations in the taste shock economy with exponential utility function. In all computations that follow, the discount rate β is equal to 0.96, and the shape parameter η is equal to 2. For the numerical simulations the upper $10^{-7}\%$ of the Pareto distribution is truncated, and the space of shocks is discretized with 600 logarithmically distributed grid points. I

⁹Note that this conclusion does not contradict the immiserization result, because more insurance will be accompanied by decreasing promised utility levels.

¹⁰To illustrate the mechanics of the argument, suppose that there are two periods and the social planner chooses two utility levels $u_1, u_2 \leq 0$ to satisfy the promise keeping constraint $-1 = u_1 + \beta u_2$, and the marginal promise keeping constraint $g = u_1 + \beta \rho u_2$. Eliminating u_2 , one gets that $u_1 = \frac{g+\rho}{1-\rho}$, and so the closer g is to zero, the larger u_1 must be.

solve the dynamic program (28) by iterating on the value function until it converges.¹¹

To check the validity of the first-order approach, one needs to be explicit about what happens when an agent reports $\hat{\theta}$ greater than her shock θ , and next period receives shock $\theta' \in [k\theta^\rho, k\hat{\theta}^\rho]$. The problem is that, along the equilibrium path, those shocks have zero probability density, and the efficient allocation does not specify what happens if the agent reports them. Since this information is needed to compute the deviator's continuation utility (22), one needs to extend the functions $u(g, \theta)$, $w'(g, \theta)$, and $h(g, \theta)$ for the off-equilibrium values $\theta < k$ in a way that does not violate incentive compatibility. There is no unique way of doing so, and any functions that satisfy the envelope condition for $\theta < k$ form a suitable candidate. I do so, and compute the sufficiency conditions for all $g \geq g^*$. I find that they hold for a tolerance criterion of $\epsilon/100$, the sufficiency conditions hold.¹²

Figure 2 shows the value function $v(g)$ for various values of the shock persistency parameter ρ , and, to make them comparable, plots them against g/ρ , rather than g . The cost minimizing marginal promised utility g^* is shown to be decreasing in ρ . When one increases ρ above 0.75, g^* increases further, but it does not appear to converge to -1 , which is the value one obtains when $\rho = 1$. Thus, the efficient allocation for ρ close to one appears to be very different from the efficient autarchic allocation when ρ is one.

In what follows, the analysis will focus on how the shock persistency affects the intratemporal and intertemporal insurance against the shocks. The intratemporal insurance is measured by the correlation between consumption and logarithm of the current shock. This statistics is equal to one in a first best allocation, to zero in autarchy, and so one can easily compare the obtained values to those two benchmarks.

¹¹ The convergence criterion ϵ is set equal to 1^{-4} . For comparison, recall that the initial promised utility is -1 , and that the average last period shock is 1.

¹²The validity of the first order approach is sometimes violated for some $g < g^*$. That does not matter because the efficient allocation always lies in the ergodic set $[g^*, 0)$.

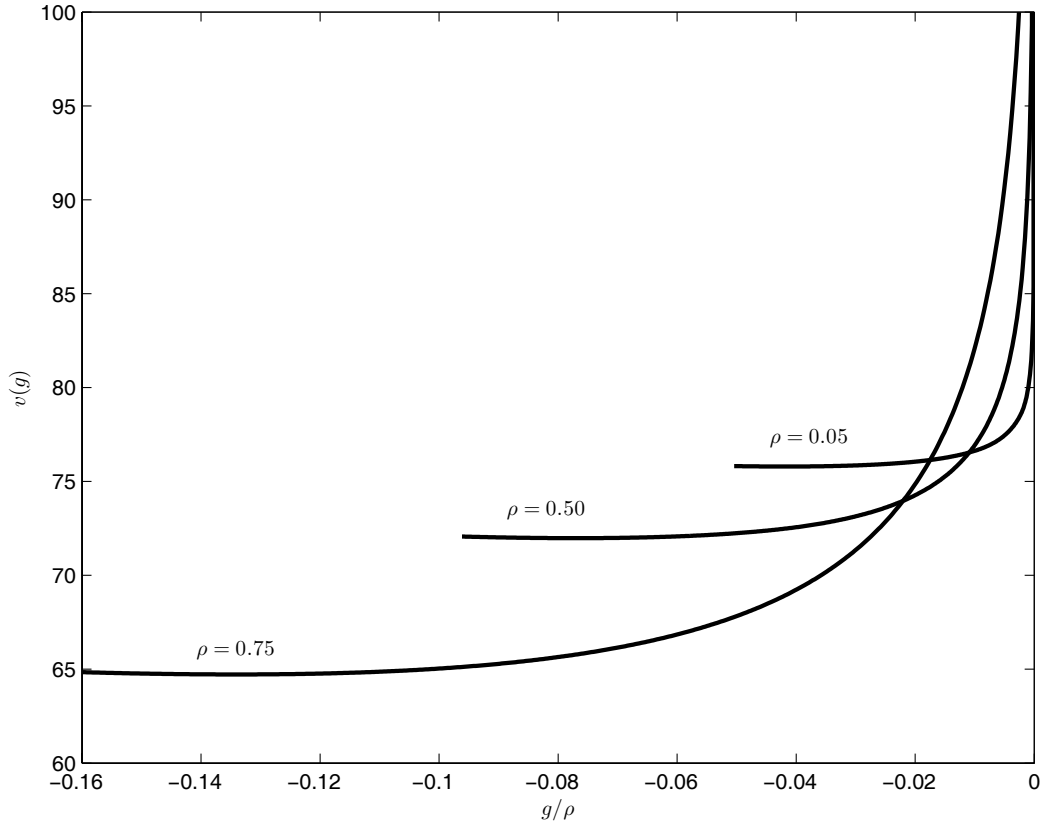


Figure 2: Value Function $v(g)$

The intertemporal insurance is measured somewhat more loosely by how fast the continuation utility disperses in the next period or over time.

Figure 3 examines how the intratemporal insurance depends on the persistence parameter ρ . It plots the correlation between current period consumption and logarithm of the shock at the cost minimizing promised utility g^* . The graph shows that the intratemporal insurance decreases with the persistency of shocks. It also shows that the degree of insurance against the shocks is overall very high, and very different from the autarchic allocation. For all computed values of ρ (between 0.01 and 0.95), it is higher than 0.999, and for ρ smaller than 0.5, it is indistinguishable from one (for an economy with i.i.d. shocks, the correlation is equal to 0.9999997).

Figure 4 plots the dynamics of the continuation utility over the first 50 periods in

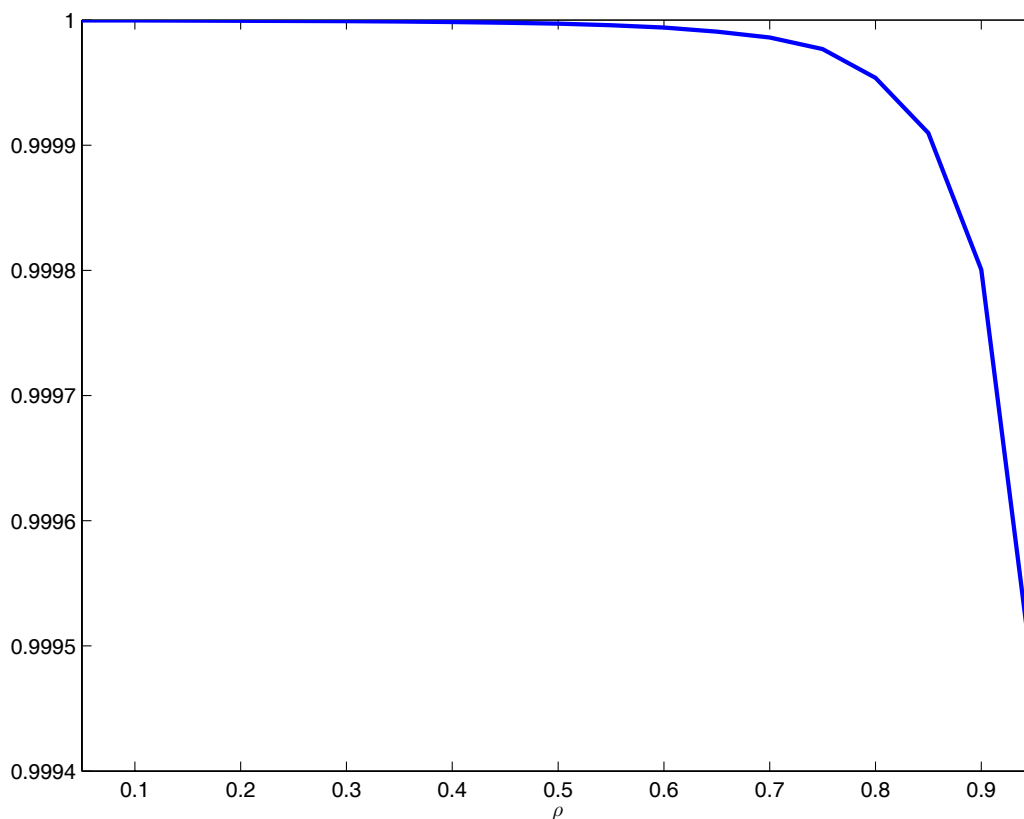


Figure 3: Correlation of Consumption and Current Shock

an economy with i.i.d. shocks, and in an economy with $\rho = 0.75$. The continuation utility is shown for two agents. The first agent happens to receive the lowest possible shock sequence, which is (k, k, k, \dots) in case of i.i.d. shocks, and $(k, k^{1+\rho}, k^{1+\rho+\rho^2}, \dots)$ in case of persistent shocks. The second agent happens to always receive the average shock sequence, which is $(1, 1, 1, \dots)$ in both cases. For the first agent the continuation utility increases over time in both economies. Similarly, for the second agent the continuation utility always decreases. The difference is that with persistent shocks the continuation utility increases or decreases much faster than when the shocks are i.i.d. That is, the continuation utility disperses faster when the shocks are persistent and the intertemporal insurance is weaker.

Figure 5 shows the current utility u and the continuation utility w' for selected

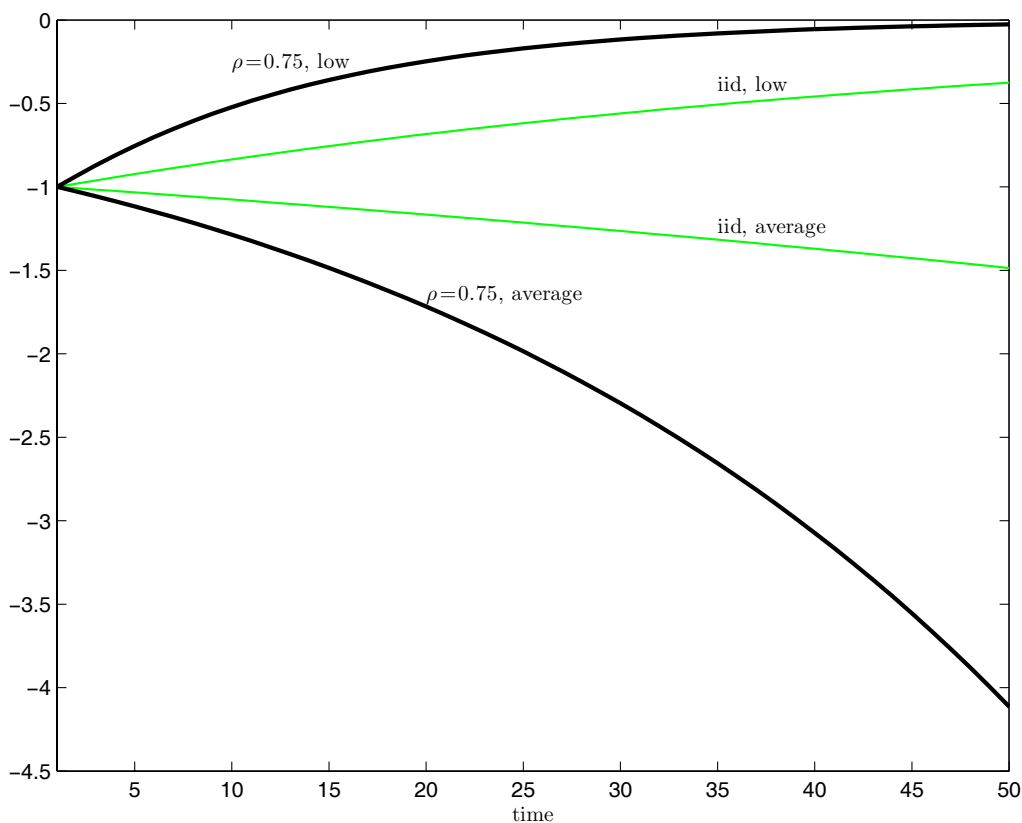


Figure 4: Dynamics of Continuation Utility

values of g , in an economy with for $\rho = 0.75$, and compares them with their counterparts in an economy with i.i.d. shocks. The general insurance principle is the same in both economies: higher current shock increases current utility because the marginal utility of consumption increases. In exchange, the continuation utility decreases. The figure shows that for $g = g^*$, the current utility allocation is very similar to the current utility allocation in an i.i.d. economy. On the other hand, the continuation utility decreases much faster with the current shock when there is persistency in shocks.

As discussed previously, for higher values of g the social planner likes to front load the resources by providing more utility in the current period. Thus, the current period utility is relatively high. To preserve incentive compatibility, the continuation utility must be relatively low for low values of θ , which is where the shock is most

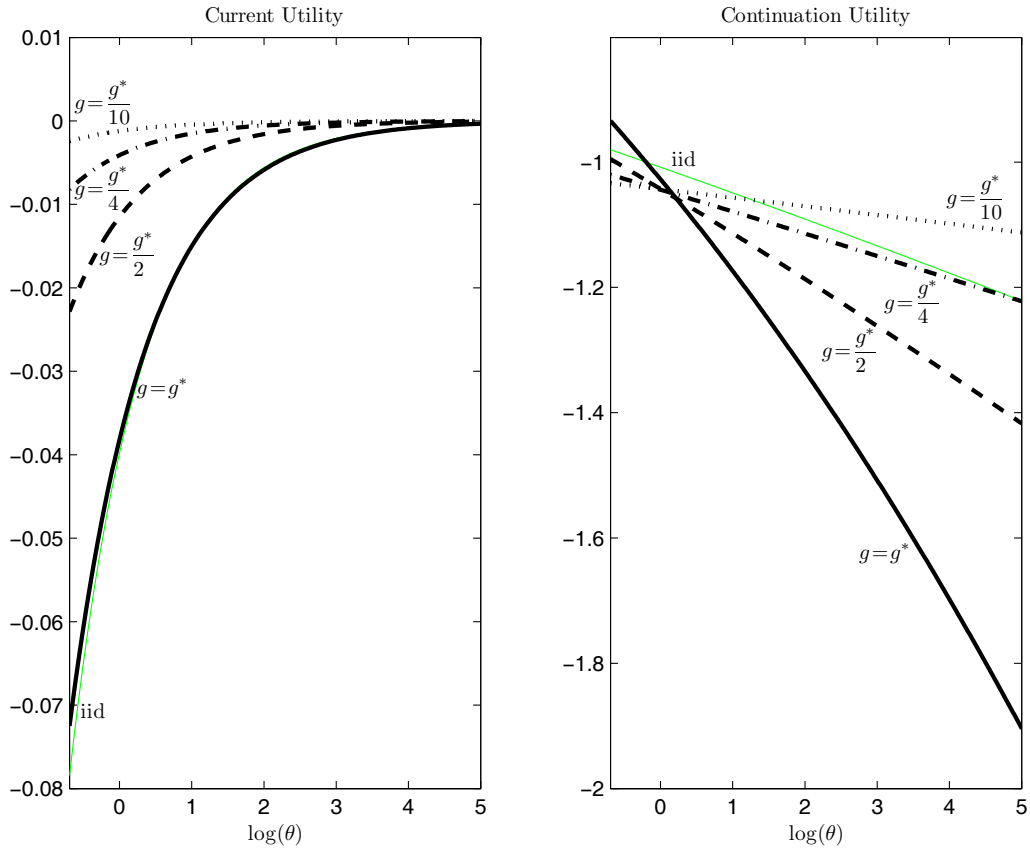


Figure 5: Current and Continuation Utility, $\rho = 0.75$

likely to be next period.¹³ Both the current utility and the continuation utility tends to be less sensitive to the current shock when g increases, and so more insurance, both intratemporal and intertemporal, is provided. Since the marginal promised utility can only increase after the first period, both types of insurance will increase over time as well.

¹³To get a sense of the underlying probability distribution, note that the probability that $\log(\theta) < 1$ is 96.6%, and that $\log(\theta) < 2$ is 99.5%.

8 Conclusions

The contribution of this paper is twofold. First, it develops a method of solving dynamic private information models with persistent shocks that is simple enough to be solved numerically, even for a continuum of shocks. It shows that, as long as the first order approach is valid, one only needs to keep track of the continuation utility of the truthtelling agent and her marginal continuation utility, rather than keeping track of the continuation utility of all possible deviating agents. I present conditions that verify validity of the first order approach and that can be checked numerically.

Second, the paper quantitatively analyzes the efficient allocations in a taste shock economy with exponential utility and Pareto distributed shocks for various values of the persistency of shocks. The results show that economies with higher persistency of shocks tend to have less intratemporal and intertemporal insurance. At the same time, since the marginal promised utility can only increase over time, the least amount of insurance is provided in the first period. The numerical simulations also show that the efficient allocations for an economy with random walk (an autarchy) are a very poor approximation for the efficient allocations in an economy with high persistency of shocks.

While the taste shock economy is a very special one, it is likely that the insurance will work in a similar way in more general Mirrleesian economies with persistent productivity shocks and endogenous labor supply. But, obviously, more quantitative studies are needed before any of the conclusions of this paper can be generalized. The first-order approach developed in this paper makes such studies tractable.

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Appendix

Proof of Lemma (1). By definition of the set \mathcal{U}^* , for any function $W' \in \mathcal{U}^*$ there must exist some functions $C : \Theta \rightarrow \mathbb{R}_+$, $Y : \Theta \rightarrow \mathbb{R}_+$, $W'' : \Theta^2 \rightarrow \mathbb{R}$, such that for any $\hat{\theta} \in \Theta$,

$$W'(\hat{\theta}, \theta) = \int_{\Theta} [U(C(\theta'), Y(\theta'), \theta') + \beta W''(\theta', \theta')] Q(\theta, d\theta') \quad (29)$$

and $W''(\theta, \cdot) \in \mathcal{U}^*$ for all $\theta \in \Theta$. The derivative of the left-hand side is equal to

$$\begin{aligned} \frac{\partial}{\partial \theta} W'(\hat{\theta}, \theta) &= \lim_{\varepsilon \rightarrow 0} \frac{W'(\hat{\theta}, \theta + \varepsilon) - W'(\hat{\theta}, \theta)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Theta} [U(C(\theta'), Y(\theta'), \theta') + \beta W''(\theta', \theta')] \frac{Q(\theta + \varepsilon, d\theta') - Q(\theta, d\theta')}{\varepsilon}. \end{aligned}$$

Both U and W'' are bounded by an integrable function. In addition, Q is differentiable with respect to its first argument and so $\lim_{\varepsilon \rightarrow 0} \frac{Q(\theta + \varepsilon, d\theta') - Q(\theta, d\theta')}{\varepsilon} = Q_1(\theta, d\theta')$. The Lebesgue Dominated Convergence Theorem then implies that

$$\frac{\partial}{\partial \theta} W'(\hat{\theta}, \theta) = \int_{\Theta} [U(C(\theta'), Y(\theta'), \theta') + \beta W''(\theta', \theta')] Q_1(\theta, d\theta').$$

Thus, $W'(\hat{\theta}, \theta)$ is differentiable in θ . Finally, since Q_1 , U and W'' are all bounded by an integrable function, $\frac{\partial}{\partial \theta} W'(\hat{\theta}, \theta)$ is also bounded by an integrable function. \square

Proof of Theorem (2). Let $R(\hat{\theta}, \theta) = U(C(\hat{\theta}), Y(\hat{\theta}), \theta) + \beta W'(\hat{\theta}, \theta)$. Let $R^*(\theta) = R(\theta, \theta)$. One has, for $\theta > 0$ and for $\hat{\theta} \in \Theta$,

$$\frac{\partial}{\partial \theta} R(\hat{\theta}, \theta) = U_{\theta}(C_t(\hat{\theta}), Y_t(\hat{\theta})) + \frac{\partial}{\partial \theta} W'(\hat{\theta}, \theta).$$

It follows from Lemma (1) that the derivative is correctly defined. Take $\theta_0 > \underline{\theta}$. For $\theta \geq \theta_0$ the expression is bounded. The first term U_{θ} is bounded because utility is bounded and linear in θ . The second term is bounded by an integrable function by Lemma (1). By the mean value theorem, $\alpha_{\eta}(\theta) = \frac{1}{\eta} [R(\theta, \theta + \eta) - R^*(\theta)]$ is also bounded

by an integrable function. Because $\lim_{\eta \rightarrow 0} \alpha_\eta(\theta) = \frac{\partial}{\partial \theta} R(\hat{\theta}, \theta)$, Lebesgue Dominated Convergence Theorem implies that for $\theta \geq \theta_0$,

$$\lim_{\eta \rightarrow 0} \int_{\theta_0}^{\theta} \alpha_\eta(\varepsilon) d\varepsilon = \int_{\theta_0}^{\theta} \lim_{\eta \rightarrow 0} \alpha_\eta(\varepsilon) d\varepsilon = \int_{\theta_0}^{\theta} \frac{\partial}{\partial \theta} R(\varepsilon, \varepsilon) d\varepsilon.$$

Hence

$$\begin{aligned} \eta \int_{\theta_0}^{\theta} \alpha_\eta(\varepsilon) d\varepsilon &= \int_{\theta_0}^{\theta} [R(\varepsilon, \varepsilon + \eta) - R^*(\varepsilon)] d\varepsilon \\ &\leq \int_{\theta_0}^{\theta} [R^*(\varepsilon + \eta) - R^*(\varepsilon)] d\varepsilon = \int_0^{\eta} [R^*(\theta + \tilde{\eta}) - R^*(\theta_0 + \tilde{\eta})] d\tilde{\eta}. \end{aligned}$$

where the inequality follows from the fact that, by incentive compatibility, $R^*(\theta + \eta) \geq R(\theta, \theta + \eta)$ and the last equality follows from the fact that integration over the interval $[\eta, \theta]$ cancels out.

Take now limits for η converging to 0 from the left and for η converging to 0 from the right. The limits must satisfy

$$\lim_{\eta \rightarrow 0^+} \frac{1}{\eta} \int_0^{\eta} [R^*(\theta + \tilde{\eta}) - R^*(\theta_0 + \tilde{\eta})] d\tilde{\eta} \geq \lim_{\eta \rightarrow 0} \int_{\theta_0}^{\theta} \alpha_\eta(\varepsilon) d\varepsilon \geq \lim_{\eta \rightarrow 0^-} \frac{1}{\eta} \int_0^{\eta} [R^*(\theta + \tilde{\eta}) - R^*(\theta_0 + \tilde{\eta})] d\tilde{\eta}.$$

The inequalities follow from the fact that η has different signs in the first and third terms. But both these terms converge to $R^*(\theta) - R^*(\theta_0)$ and so

$$R^*(\theta) - R^*(\theta_0) = \int_{\theta_0}^{\theta} \frac{\partial}{\partial \theta} R(\varepsilon, \varepsilon) d\varepsilon.$$

Letting $\theta_0 \rightarrow \underline{\theta}$ concludes the proof. \square

Proof of Theorem (3). Let $R(\hat{\theta}, \theta) = U(C(\hat{\theta}), Y(\hat{\theta}), \theta) + \beta W'(\hat{\theta}, \theta)$. Suppose that $\frac{\partial}{\partial \theta} R(\hat{\theta}, \theta)$ is nondecreasing in $\hat{\theta}$ for all θ . Assume that $\hat{\theta} < \theta$.

$$R^*(\theta) - R^*(\hat{\theta}) = \int_{\hat{\theta}}^{\theta} \frac{\partial}{\partial \theta} R(\varepsilon, \varepsilon) d\varepsilon \geq \int_{\hat{\theta}}^{\theta} \frac{\partial}{\partial \theta} R(\hat{\theta}, \varepsilon) d\varepsilon = R(\hat{\theta}, \theta) - R^*(\hat{\theta})$$

where the first equality follows from the fact that the necessary envelope condition is supposed to hold. Similar arguments apply for $\hat{\theta} > \theta$ and so one gets $R^*(\theta) \geq R(\hat{\theta}, \theta)$ for all $\hat{\theta} \in \Theta$. \square

Proof of Theorem (5). It will be first shown that $v \leq V$. The reverse inequality is shown next. Both inequalities together imply that $v = V$.

a) Suppose that Z is the efficient allocation rule. Define a relaxed allocation rule z recursively as follows. For the initial period set

$$\begin{aligned} c_1(w, \theta_0, \theta_1) &= C_1(w, \theta_0, \theta_1), \\ y_1(w, \theta_0, \theta_1) &= Y_1(w, \theta_0, \theta_1), \\ w_2(w, \theta_0, \theta_1) &= W_2(w, \theta_0, \theta_1, \theta_1), \\ g_2(w, \theta_0, \theta_1) &= \frac{\partial}{\partial \theta} W_2(w, \theta_0, \theta_1, \theta_1). \end{aligned}$$

Let $\mathbf{W}_2(\cdot) = W_2(w, \theta_0, \theta_1, \cdot)$, and for $t > 1$, $\mathbf{W}_{t+1}(\cdot) = W_{t+1}(\mathbf{W}_t(\cdot), \theta_-, \theta, \cdot)$. Then define a relaxed allocation rule as follows: If $w_t = \mathbf{W}_t(\theta_-)$ and $g_t = \frac{\partial}{\partial \theta} \mathbf{W}_t(\theta_-)$, let

$$\begin{aligned} c_t(w_t, g_t, \theta_-, \theta) &= C_t(\mathbf{W}_t(\cdot), \theta_-, \theta) \\ y_t(w_t, g_t, \theta_-, \theta) &= Y_t(\mathbf{W}_t(\cdot), \theta_-, \theta) \\ w_{t+1}(w_t, g_t, \theta_-, \theta) &= W_{t+1}(\mathbf{W}_t(\cdot), \theta_-, \theta, \theta) \\ g_{t+1}(w_t, g_t, \theta_-, \theta) &= \frac{\partial}{\partial \theta} W_{t+1}(\mathbf{W}_t(\cdot), \theta_-, \theta, \theta). \end{aligned}$$

If $w_t \neq \mathbf{W}_t(\theta_-)$ or $g_t \neq \frac{\partial}{\partial \theta} \mathbf{W}_t(\theta_-)$, and $(w_t, g_t) \in \mathcal{V}^*(\theta_-)$, pick any functions c_t, y_t, w_{t+1} and g_{t+1} that satisfy (11), (12), (13) and (14). Such functions exist since $(w_t, g_t) \in \mathcal{V}^*(\theta_-)$.

The relaxed allocation rule satisfies the promised keeping constraint (12) and the marginal promise keeping constraint (13) because Z satisfies the promise keeping constraint (1). It also satisfies the envelope condition (11), because Z satisfies the incentive compatibility constraint (3). Finally, since Z satisfies (4), z satisfies (14). One can also show that the present value of costs of the relaxed allocation rule is given by V . All these results are fairly straightforward to show, and the proofs are omitted. Those results imply that $v \leq V$.

b) In order to reduce notational burden, define the following functions:

$$R_t(\mathbf{w}(\cdot), \theta_-, \hat{\theta}, \theta) = U(C_t(\mathbf{w}(\cdot), \theta_-, \hat{\theta}), Y_t(\mathbf{w}(\cdot), \theta_-, \hat{\theta}), \theta) + \beta W_{t+1}(\mathbf{w}(\cdot), \theta_-, \hat{\theta}, \theta).$$

$$r_t(w, g, \theta_-, \hat{\theta}, \theta) = U(c_t(w, g, \theta_-, \hat{\theta}), y_t(w, g, \theta_-, \hat{\theta}), \theta) + \beta w_{t+1}(w, g, \theta_-, \hat{\theta}).$$

Suppose now that z is the efficient relaxed allocation rule. Define an allocation rule Z as follows: In the first period, let

$$C_1(w, \theta_0, \theta_1) = c_1(w, \theta_0, \theta_1)$$

$$Y_1(w, \theta_0, \theta_1) = y_1(w, \theta_0, \theta_1)$$

$$W_2(w, \theta_0, \theta_1, \cdot) = \int_{\Theta} r_2(w_2(w, \theta_0, \theta_1), g_2(w, \theta_0, \theta_1), \theta', \theta') Q(\cdot, d\theta').$$

For $t > 1$, for all $(w, g) \in \mathcal{V}^*(\theta_-)$, let $\mathbf{w}_t(w, g, \theta_-, \cdot) = \int_{\Theta} r_t(w, g, \theta_-, \theta, \theta) Q(\cdot, d\theta)$.

Then define

$$C_t(\mathbf{w}_t(w, g, \cdot), \theta_-, \theta) = c_t(w, g, \theta_-, \theta)$$

$$Y_t(\mathbf{w}_t(w, g, \cdot), \theta_-, \theta) = y_t(w, g, \theta_-, \theta)$$

$$W_{t+1}(\mathbf{w}_t(w, g, \cdot), \theta_-, \theta, \cdot) = \mathbf{w}_{t+1}(w_{t+1}(w, g, \theta_-, \theta), g_{t+1}(w, g, \theta_-, \theta), \theta, \cdot)$$

For any $\mathbf{w}_t(\cdot) \neq \mathbf{w}_t(w, g, \cdot)$ such that $\mathbf{w}_t(\cdot) \in \mathcal{U}^*$, pick any functions C_t, Y_t and W_{t+1} that satisfy (1), (3) and (4). Such functions exist since $\mathbf{w}_t(\cdot) \in \mathcal{U}^*$.

It needs to be verified that Z satisfies the promise keeping constraint (1), the incentive compatibility constraint (3), and that the implementability constraint (4) holds. In what follows, the proof is provided for $\mathbf{w}_t(\cdot) = \mathbf{w}_t(w, g, \cdot)$, $t > 0$, since the remaining cases are trivial. Notice first that

$$\begin{aligned} & R_t(\mathbf{w}_t(w, g, \theta_-, \cdot), \theta_-, \theta, \theta) \\ &= U(c_t(w, g, \theta_-, \theta), y_t(w, g, \theta_-, \theta) + \beta \mathbf{w}_{t+1}(w_{t+1}(w, g, \theta_-, \theta), g_{t+1}(w, g, \theta_-, \theta), \theta, \theta)) \\ &= U(c_t(w, g, \theta_-, \theta), y_t(w, g, \theta_-, \theta) + \beta w_{t+1}(w, g, \theta_-, \theta)) \\ &= r_t(w, g, \theta_-, \theta, \theta), \end{aligned} \tag{30}$$

because $\mathbf{w}_t(w, g, \theta_-, \theta) = w$ by construction.

b-i) Z satisfies the promise keeping constraint (1). We have,

$$\begin{aligned} \int_{\Theta} R_t(\mathbf{w}_t(w, g, \theta_-, \cdot), \theta_-, \theta, \theta) Q(\cdot, d\theta) &= \int_{\Theta} r_t(w, g, \theta_-, \theta, \theta) Q(\cdot, d\theta) \\ &= \mathbf{w}_t(w, g, \theta_-, \cdot), \end{aligned}$$

where the first equality follows from (30) and the second equality follows from definition of \mathbf{w}_t .

b-ii) Z satisfies the incentive compatibility constraint (3). Note first that

$$\begin{aligned} R_t(\mathbf{w}_t(w, g, \theta_-, \cdot), \theta_-, \theta, \theta) &= r_t(w, g, \theta_-, \theta, \theta) \\ &= \int_{\underline{\theta}}^{\theta} [U_{\theta}(c_t(w, g, \theta_-, \varepsilon), y_t(w, g, \theta_-, \varepsilon)) + \beta g_{t+1}(w, g, \theta_-, \varepsilon)] d\varepsilon + w_0 \\ &= \int_{\underline{\theta}}^{\theta} [U_{\theta}(C_t(\mathbf{w}_t(w, g, \cdot), \theta_-, \varepsilon), Y_t(\mathbf{w}_t(w, g, \theta_-, \cdot), \theta_-, \varepsilon)) \\ &\quad + \beta \frac{\partial}{\partial \theta} W_{t+1}(\mathbf{w}_t(w, g, \cdot), \theta_-, \varepsilon, \varepsilon)] d\varepsilon + w_0. \end{aligned}$$

The first equality follows from (30), second one follows from the fact that the envelope condition (11) holds, and the third one follows from the definition of C_t , Y_t and W_{t+1} . Hence Z satisfies (9). Second, note that

$$\begin{aligned} &U_{\theta}(c_t(w, g, \theta_-, \hat{\theta}), y_t(w, g, \theta_-, \hat{\theta})) \\ &\quad + \beta \int_{\Theta} r_{t+1}(w_{t+1}(w, g, \theta_-, \hat{\theta}), g_{t+1}(w, g, \theta_-, \hat{\theta}), \hat{\theta}, \theta', \theta') Q_1(\theta, d\theta') \\ &= U_{\theta}(c_t(w, g, \theta_-, \hat{\theta}), y_t(w, g, \theta_-, \hat{\theta})) \\ &\quad + \beta \frac{\partial}{\partial \theta} \mathbf{w}_{t+1}(w_{t+1}(w, g, \theta_-, \hat{\theta}), g_{t+1}(w, g, \theta_-, \hat{\theta}), \hat{\theta}, \theta) \\ &= U_{\theta}(C_t(\mathbf{w}_t(w, g, \theta_-, \cdot), \theta_-, \theta), Y_t(\mathbf{w}_t(w, g, \theta_-, \cdot), \theta_-, \theta), \theta) \\ &\quad + \beta \frac{\partial}{\partial \theta} W_{t+1}(\mathbf{w}_t(w, g, \cdot), \theta_-, \hat{\theta}, \theta). \end{aligned} \tag{31}$$

The first equality follows from the definition of \mathbf{w}_{t+1} , and the second equality follows from the definition of C_t , Y_t and W_{t+1} . Since the first order approach is valid, the

first expression in (31) is increasing in $\hat{\theta}$ for all $\theta \in \Theta$, and so is the last expression. But the last expression is equal to (10). Hence Z satisfies both (9) and (10) and so is incentive compatible.

b-iii) Z satisfies the implementability constraint (4). Fix $T < \infty$ and let $\mathcal{U}_T^T = \mathcal{L}$. For $t = 1 \dots T$ define

$$\mathcal{U}_{t-1}^T = \{\mathbf{w}(\cdot) \in \mathcal{L} : \exists C : \Theta \rightarrow \mathbb{R}_+, Y : \Theta \rightarrow \mathbb{R}_+, W' : \Theta^2 \rightarrow \mathbb{R}$$

s.t.

$$\mathbf{w}(\cdot) = \int_{\Theta} [U(C(\theta), Y(\theta), \theta) + \beta W'(\theta, \theta)] Q(\cdot, d\theta)$$

$$U(C(\theta), Y(\theta), \theta) + \beta W'(\theta, \theta) \geq U(C(\hat{\theta}), Y(\hat{\theta}), \theta) + \beta W'(\hat{\theta}, \theta) \quad \forall (Y(\hat{\theta}), \theta) \in \mathcal{Y}$$

$$W'(\theta, \cdot) \in \mathcal{U}_t^T\}.$$

It follows from parts *b-i)* and *b-ii)* that $W_T(\mathbf{w}_{t-1}(w, g, \cdot), \theta_-, \theta, \cdot) \in \mathcal{U}_{T-1}^T$. An induction argument implies that $W_{t+1}(\mathbf{w}_t(w, g, \cdot), \theta_-, \theta, \cdot) \in \mathcal{U}_t^T$ for $t < T - 1$. The proof is complete when one shows that $\lim_{T \rightarrow \infty} \mathcal{U}_t^T = \mathcal{U}^*$ for all t . To show this, note first that, by construction, $\mathcal{U}_t^T = \mathcal{U}_1^{T-t+1}$. Thus, it is enough to show that $\lim_{T \rightarrow \infty} \mathcal{U}_1^T = \mathcal{U}^*$. This follows from the fact that \mathcal{U}^* is a fixed point of (5).

The allocation rule Z can be shown to have the same present value of costs as the efficient relaxed allocation rule z , given by v . Since Z is not necessarily efficient, we have $V \leq v$. □

Proof of Lemma (6). Ignoring the envelope constraint (19) and using the fact that current period utility can always be chosen to be report independent, the lower bound-

ary $\underline{\mathcal{V}}(g, \theta)$ satisfies the following Bellman equation:

$$\underline{\mathcal{V}}(g, \theta_-) = \min_{u, g'} u\theta_-^\rho + \beta \int_{k\theta_-^\rho} \underline{\mathcal{V}}(g'(\theta), \theta) f(\theta_-, d\theta) \quad (32)$$

$$s.t. \quad u \in [\underline{u}, \bar{u}]$$

$$g'(\theta) \in [\underline{g}(\theta), \bar{g}(\theta)] \quad (33)$$

$$\frac{\theta_-}{\rho} g = u\theta_-^\rho + \beta \int_{k\theta_-^\rho}^\infty \theta g'(\theta_-) f(\theta_-, \theta) d\theta \quad (34)$$

The proof uses the guess and verify method. Suppose that $\underline{\mathcal{V}}(g, \theta_-)$ takes the form suggested by the Lemma, i.e. that for $i \geq 1$,

$$\underline{\mathcal{V}}(g, \theta_-) = \frac{\theta_-}{\rho^i} [g - \underline{g}_{i-1}(\theta_-)] + \underline{w}_{i-1}(\theta_-), \quad g \in [\underline{g}_{i-1}(\theta_-), \underline{g}_i(\theta_-)].$$

Assume that $g \in [\underline{g}(\theta_-), \underline{g}_1(\theta_-)]$. Then $u < \bar{u}$, otherwise the right-hand side of (34) is at least $\underline{g}_1(\theta_-)$, and so (34) cannot hold. Therefore, the Lagrange Multiplier on (34) satisfies $\lambda(g) \leq 1$. The first order condition in $g'(\theta)$ implies that if (33) does not bind then $\underline{\mathcal{V}}_g(g'(\theta), \theta) = \lambda\theta$. Hence $\underline{\mathcal{V}}_g(g'(\theta), \theta) \leq \theta$. But since $\underline{\mathcal{V}}_g(g'(\theta), \theta) \geq \frac{\theta}{\rho} > \theta$ by the guess, the continuation marginal promised utility must be the lowest possible, $g'(\theta) = \underline{g}(\theta)$. The current period utility u is then determined as a residual from (34). Substituting the solution to (32) yields

$$\begin{aligned} \underline{\mathcal{V}}(g, \theta_-) &= \frac{\theta_-}{\rho} g + \beta \int_{k\theta_-^\rho}^\infty [\underline{\mathcal{V}}(\underline{g}(\theta), \theta) - \theta \underline{g}(\theta)] f(\theta_-, d\theta) \\ &= \frac{\theta_-}{\rho} g + \beta \int_{k\theta_-^\rho}^\infty [\underline{w}(\theta) - \theta \underline{g}(\theta)] f(\theta_-, d\theta) \\ &= \frac{\theta_-}{\rho} [g - \underline{g}(\theta_-)] + u\theta_-^\rho + \beta \int_{k\theta_-^\rho}^\infty \underline{w}(\theta) f(\theta_-, d\theta) \\ &= \frac{\theta_-}{\rho} [g - \underline{g}(\theta_-)] + \underline{w}(\theta_-), \end{aligned}$$

when $g \in [\underline{g}(\theta_-), \underline{g}_1(\theta_-)]$. It is easy to see that $\underline{\mathcal{V}}(\underline{g}(\theta_-), \theta_-) = \underline{w}(\theta_-)$ and that $\lim_{g \rightarrow \underline{g}_1(\theta_-)} \underline{\mathcal{V}}(g, \theta_-) = \underline{w}_1(\theta_-)$. Thus, the guess is verified for $g < \underline{g}_1(\theta_-)$.

Now assume that the proof is verified for $g < \underline{g}_I(\theta_-)$, for some $I \geq 1$. Let $g \in [\underline{g}_I(\theta_-), \underline{g}_{I+1}(\theta_-)]$. The Lagrange multiplier $\lambda(g)$ satisfies $\lambda(g) > 1$, since otherwise

$g'(\theta) = \underline{g}(\theta)$ (for the same reason as before), and (34) cannot hold. But $\lambda > 1$ implies that $u = \bar{u}$. This in turn yields that $g'(\theta) \in [\underline{g}_{I-1}(\theta), \underline{g}_I(\theta)]$, otherwise (34) cannot hold. In order to have that, the Lagrange multiplier must satisfy $\lambda = \frac{1}{\rho^I}$. It follows that any $g'(\theta) \in [\underline{g}_{I-1}(\theta), \underline{g}_I(\theta)]$ satisfies the first order condition in $g'(\theta)$. In order to satisfy (34), choose the following solution:

$$g'(g, \theta) = \zeta_I(g) \underline{g}_{I-1}(\theta) + (1 - \zeta_I(g)) \underline{g}_I(\theta)$$

where $\zeta_I(g) = \frac{\underline{g}_{I+1}(\theta_-) - g}{\underline{g}_{I+1}(\theta_-) - \underline{g}_I(\theta_-)}$. The value function in the interval $g \in [\underline{g}_I(\theta_-), \underline{g}_{I+1}(\theta_-)]$ is then equal to

$$\begin{aligned} \underline{\mathcal{V}}(g, \theta_-) &= \bar{u}\theta_-^\rho + \beta \int_{k\theta_-^\rho} \underline{\mathcal{V}}(g'(g, \theta), \theta) f(\theta_-, d\theta) \\ &= \bar{u}\theta_-^\rho + \beta \int_{k\theta_-^\rho} \frac{\theta}{\rho^i} [g'(g, \theta) - \underline{g}_{I-1}(\theta)] + \underline{w}_{I-1}(\theta) (\theta_-, d\theta) \\ &= \frac{\beta}{\rho^i} \int_{k\theta_-^\rho} [\theta g'(g, \theta) - \theta \underline{g}_{I-1}(\theta)] (\theta_-, d\theta) + \underline{w}_I(\theta_-) \\ &= \frac{\beta}{\rho^i} (1 - \zeta_I(g)) \int_{k\theta_-^\rho} [\theta \underline{g}_I(\theta) - \theta \underline{g}_{I-1}(\theta)] (\theta_-, d\theta) + \underline{w}_I(\theta_-) \\ &= \frac{\theta_-}{\rho^{i+1}} (1 - \zeta_I(g)) [\underline{g}_{I+1}(\theta_-) - \underline{g}_I(\theta_-)] + \underline{w}_I(\theta_-) \\ &= \frac{\theta_-}{\rho^{i+1}} [g - \underline{g}_I(\theta_-)] + \underline{w}_I(\theta_-). \end{aligned}$$

It is easy to see that $\underline{\mathcal{V}}(\underline{g}_I(\theta_-), \theta_-) = \underline{w}_I(\theta_-)$ and that $\lim_{g \rightarrow \underline{g}_{I+1}(\theta_-)} \underline{\mathcal{V}}(g, \theta_-) = \underline{w}_{I+1}(\theta_-)$. The guess is therefore verified for $g < \underline{g}_{I+1}(\theta_-)$. \square