Lecture 2—Time value of Money

Net Present Value/Time Value of Money

Basic idea: Money received now is worth more than money in the future.

Why?

Example

We can invest in risk-free investments and gain money. For example, say a one year T-bill has a return of 2.8%.

What will \$1000 invested in the bond return at the end of one year?

$$\$1000+(0.028 \; x \; \$1000) = \$1000 \; x \; 1.028 = \$1028$$

In general

How much would we have to invest (at rate *r*) to obtain C_1 in one year?

From the formula for interest,

$$C_1 = C_0 \, x \, (1+r)$$

Where C_0 is the principal (original investment).

Present Value:

Solving algebraically for C_0:

$$C_0 = \frac{C_1}{(1 + r)}$$

C_0 then is the ***Present Value*** of obtaining the payout C_1 in one year.

Multiple years Future Value

The money in our account after one year is

$$C_1 = C_0 \cdot (1 + r)$$

C_1 then becomes our 'principal' for the next year's compounding

In general, $$C_n = C_0 \cdot (1 + r)^n$$

Where C_n is the *Future Value* of C_0 in n years.

Examples:

- Suppose we invest $100 in a T-bill that returns 6% per year. How much would we have after 1 year? After 3 years? After 30 years?

Examples (continued)

- 2008 Tax rebate: In mid-2008 President Bush authorized a tax cut and rebate to be given to taxpayers. Couples received up to $1200, and single people with no dependents $600.

- This was done to stimulate the economy. Was this effective?

Assumptions:

- Tax rebate of $1200 received April.
- Interest rate is 6%.
- That money will need to be repaid at some point, so assume that $1236 (using 3% interest for gov’t borrowing) will need to be repaid the following April.

- How much is this tax rebate worth? Will this drastically increase spending?

Actual results (from a survey)

- “In 2001, many households received rebate checks as advanced payments of the benefit of the new, 10 percent federal income tax bracket. A survey conducted at the time the rebates were mailed finds that few households said that the rebate led them mostly to increase spending. A follow-up survey in 2002, as well as a similar survey conducted after the attacks of 9/11, also indicates low spending rates. ”

--- “**Did the 2001 Tax Rebate Stimulate Spending? Evidence from Taxpayer Surveys**”, Shapiro, Matthew D. and Joel B. Slemrod, NBER working paper W9308

Multi-Period Present Value?

Similar to the single period case, solve for C_0. Since

$$C_n = C_0 \cdot (1 + r)^n$$

the Present Value of C_n will be

$$C_0 = \frac{C_n}{(1 + r)^n}$$

Aside: Compounding. Assume 5% interest

Year	Beginning	Interest	Final
1	100	5	105
2	105	5.25	110.25
3	110.25	5.51	115.76
4	115.76	5.79	121.55
5	121.55	6.08	127.63

Also, $127.63 = 100 \cdot (1.05)^5 = 100 \cdot 1.2763$. In 30 years the \$100 would grow to $100 \cdot (1.05)^{30} =$ \$432.19

Observations:

Simple 5% interest (non-compounded) would return $5 every year for 30 years. Total payout: $150.

5% interest pays $432.19.

At 12% (compounded) interest, the payout is $2995.99.

This shows the power of both the interest rate and compounding.

Another aside: Doubling time

Law of 72: Doubling time. $V_n = 2 \cdot V_0$

$$n^* = \frac{\ln 2}{\ln(1+r)} = \frac{0.69315}{\ln(1+r)}$$

This is approximately $n^* \cong {}^{72}\!/_{100r}$

For 6% interest, the law of 72 gives a doubling time of 12 years. The exact answer is 11.90 years.

Present Values to Net Present Value

Present values are additive.

Sum of PV gives ***Net Present Value*** for a single project.

Likewise, we can add different projects together.

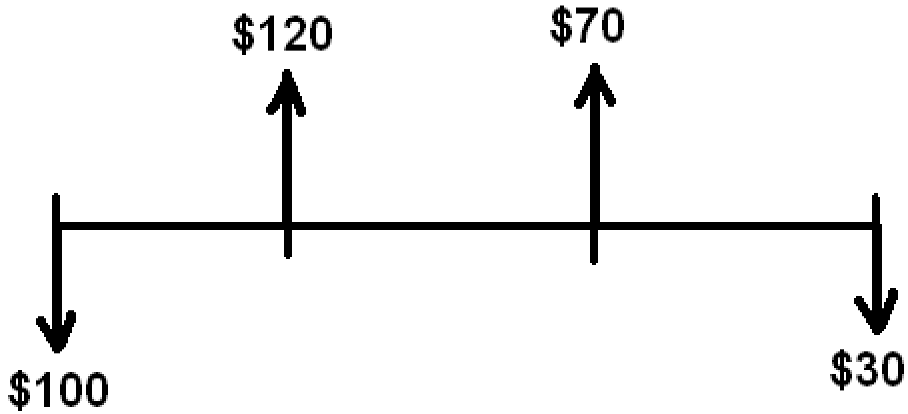
How to use NPV in Decisions

Our rule for selecting projects will be:

Projects with positive net present value should be undertaken.

If only a subset of projects can be undertaken, pick the highest NPV possible.

Example 1, from the previous lecture:



If we assume a 10% interest rate,

$$NPV = -\$100 + \frac{\$120}{1.1} + \frac{\$70}{1.1^2} - \frac{\$30}{1.1^3} = \$44.40$$

At a 60% interest rate, NPV is -$4.98

Example 2: Project 1: \$100 a year for 4 years, or
Project 2: \$500 at the end of 4 years.

NPV of Project 1:

$$NPV = \frac{100}{(1+r)} + \frac{100}{(1+r)^2} + \frac{100}{(1+r)^3} + \frac{100}{(1+r)^4}$$

NPV of Project 2:

$$NPV(2) = \frac{500}{(1+r)^4}$$

At an interest rate of r = 10%, NPV of project 1 is \$316.99.
NPV of project 2 is \$341.51.

At an interest rate of $r = 10\%$, NPV of project 1 is \$316.99. NPV of project 2 is \$341.51.

- Which project(s) should we undertake?
- If only one is possible, which project should we chose?
- If there was a cost at the start of \$330, but we could do any number of projects, which should we undertake?

Effective and Stated Interest Rates

A 10% **stated** annual interest rate, compounded semi-annually means that 5% interest is computed every six months.

What is the **effective** annual interest rate to which this corresponds?

General result with m compounding periods in one year, stated rate r:

$$EAIR = \left(1 + \frac{r}{m}\right)^{m} - 1$$

Continuous compounding:

$$m \rightarrow \infty$$

From Calculus,

$$\left(1 + r/m\right)^{m} \rightarrow e^{r}$$

The ‘power’ of compounding periods: Assume $100 in the bank for one year with a SAIR of 10%.

Compounding Period	Value
Annual	$110.00
Semi-Annual	$110.25
Quarterly	$110.38
Daily	$110.516
Continuously	$110.517

Why is continuous compounding useful in finance? The returns with continuous compounding can be simply added together, but discrete compounding rates cannot be added together.

For example:

Year	Begin	End	Simple	Cont.
2002	5000	3000	-40%	-51.08%
2003	3000	4000	+33.3%	+28.77%
Two Year	5000	4000	-20%	-22.31%

Annuities and Perpetuities

A Perpetuity gives a constant cash flow *C* at the end of every year in the future (infinitely long). The cash flow looks like:

Year	0	1	2	3	…	n	n+1	…
Cash		C	C	C	…	C	C	…

NPV is C/r.

Annuities and Perpetuities continued

The other (mathematical) way to calculate the NPV is to sum the infinite series:

$$NPV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots \quad (1)$$

$$(1+r) \cdot NPV = C + \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots \quad (2)$$

Subtracting (1) from (2),

$$r \cdot NPV = C$$

And we obtain the previous result (thankfully!)

Annuities and Perpetuities continued

Year	0	1	2	…	T	T+1	T+2	…
Perpetuity 1		C	C	…	C	C	C	…
Perpetuity 2				…		-C	-C	…
The Annuity		C	C	…	C			

$$NPV(Annuity) = \frac{C}{r} - \frac{C}{r}\left[\frac{1}{(1+r)^T}\right]$$

$$= C\left[\frac{1}{r} - \frac{1}{r(1+r)^T}\right]$$

Growing Perpetuity

If the cash flows grow at a rate *g*, and the interest rate is *r*, the cash flows will be

year	0	1	2	3	…	T	…
P'ty		C	C(1+g)	$C(1+g)^2$	…	$C(1+g)^T$	…

Using the previous algebraic method to sum this series, except we multiply by (1+*r)/(*1+*g)* to obtain equation (2'). The result, after more algebra is

$$NPV = \frac{C}{r-g}$$

Example of an Annuity

Suppose we are given an annuity that pays $500 a year for the next 30 years (the first payment is made in one year's time). If the appropriate interest rate is 6%, what is the NPV of this annuity?

What if the first payment is made today instead of one year hence?

Main ideas:

- Money now is worth more than money in the future. Why?
- Why can we add PVs to get NPV? How do we use NPV?
- Stated Interest rate needs to have compounding period defined.
- Do you need to use the Annuity/Perpetuity formulas?