Problem 2.49 old edition, 2.50 new edition

Original Problem: Someone consumes a single good $x$ and her indirect utility is

$$v(p, y) = G\left(A(p) + \frac{\bar{y}^\eta y^{1-\eta}}{1 - \eta}\right),$$

where

$$A(p) = \int_p^\phi x(\xi, \bar{y})d\xi.$$

(a) Derive the consumer’s demand for $x$ and show that it has constant income elasticity equal to $\eta$.

(b) Suppose that the consumer has income equal to $\bar{y}$ and the price of $x$ rises from $p$ to $p' > p$. Argue that the change in the consumer’s utility caused by the price change is

$$-\int_p^{p'} x(\xi, \bar{y})d\xi < 0.$$

Interpret this measure.

Critical Comment: This problem maintains that there is only one commodity, $x$. If there is only one commodity then from the budget it follows that $x(p, y) = \frac{y}{p}$ and if this is the case, the income elasticity of demand for $x$ can only be 1. Moreover, the indirect utility function proposed in the problem

$$v(p, y) = G\left(A(p) + \frac{\bar{y}^\eta y^{1-\eta}}{1 - \eta}\right)$$

is not defined in a straightforward way when $\eta = 1$.

I think this becomes a much more reasonable problem if it is recast something like this.

Proposed Alternative: There are two commodities. Someone has an indirect utility function

$$v(p_1, p_2, y) = G\left(A(p_1, p_2) + \frac{\bar{y}^\eta y^{1-\eta}}{1 - \eta}\right)$$

Assume that the price of good 2 is fixed at $\bar{p}_2$ and that

$$A(p_1, \bar{p}_2) = \int_{p_1}^{p_2} f(\xi, \bar{p}_2, \bar{y})d\xi$$

for some continuous function $f$.

A) Derive the consumer’s demand for good 1 and show that it has constant income elasticity equal to $\eta$.

B) Suppose that the consumer has income $\bar{y}$ and the price of good 1 rises from $p$ to $p'$. Show that the change in the consumer’s utility can be measured by

$$\int_p^{p'} x_1(\xi, \bar{p}_2, \bar{y})d\xi.$$
Interpret this measure.

**Answer to Part A:** Recall that

\[ x_1(p_1, p_2, y) = -\frac{\partial v(p_1, p_2, y)}{\partial p_1} - \frac{\partial v(p_1, p_2, y)}{\partial y} \]

Now

\[ \frac{\partial v(p_1, p_2, y)}{\partial p_1} = \frac{\partial A(p_1, p_2)}{\partial p_1} = -f(p_1, \bar{p}_2, \bar{y}) \]

and

\[ \frac{\partial v(p_1, p_2, y)}{\partial y} = \bar{y}^\eta - \eta. \]

Substituting into equation , we have

\[ x_1(p_1, \bar{p}_2, y) = f(p_1, \bar{p}_2, \bar{y})\bar{y}^\eta y^\eta. \]  

(1)

Then

\[ \ln x_1(p_1, \bar{p}_2, y) = \ln f(p_1, \bar{p}_2, \bar{y}) - \eta \ln \bar{y} + \eta \ln y \]

and the income elasticity of demand for the good \( x \) must be

\[ \frac{d \ln x_1(p, y)}{d \ln y} = \eta. \]

**Answer to Part B:** Define \( u^*(x) = G^{-1}(u(x)) \). Since \( G \) is an increasing function, so is \( G^{-1} \). Therefore \( u^*(x) \) represents this person’s preferences. Now

\[ u^*(x(p_1', \bar{p}_2, \bar{y})) - u^*(x(p_1, \bar{p}_2, \bar{y})) = A(p_1', \bar{p}_2) + \bar{y} - A(p_1', \bar{p}_2) - \bar{y} \]

\[ = A(p_1, \bar{p}_2) - A(p_1, \bar{p}_2) \]

\[ = -\int_{p_1}^{p_1'} f(\xi, \bar{p}_2, \bar{y}) d\xi \]

\[ = -\int_{p_1}^{p_1'} x_1(\xi, \bar{p}_2, \bar{y}) d\xi \]  

(2)

where the last equality in 2 follows from Equation 1.
Thus the difference in utility as measured by $u^*$ is equal to

$$-\int_{p_1}^{p'_1} x_1(\xi, \bar{p}_2, \bar{y}) d\xi$$

If we draw a demand curve with $p_1$ on the vertical axis and $x_1(p_1, \bar{p}_2, \bar{y})$ on the horizontal axis, then this integral is the “loss or consumer surplus”, which is the difference between the area under the demand curve above a horizontal line at height $p'$ and the area under the demand curve above a horizontal line at height $p$. This is the traditional diagram of the change in consumers’ surplus.