Some Properties of Indirect Utility.

▷ If preferences be represented by a continuous strictly increasing function defined on $\mathbb{R}^n_+$, then for any price vector $p \in \mathbb{R}_{++}$ there is a well-defined function $v(p, m)$ which is $\{\max u(x) | px \leq m\}$. (How do we know this?)

▷ $v(p, m)$ is homogeneous of degree 0 in $(p, m)$. (Prove it.)

▷ $v(p, m)$ is strictly increasing in $m$ and non-increasing in $p$. (Prove it.)

▷ $v(p, m)$ is a quasi-convex function.

▷ $v(p, m)$ is a continuous function.
**Quasi-convexity of indirect utility**

- Where $A$ is a convex subset of $\mathbb{R}^n$, a function $f : A \rightarrow \mathbb{R}$ is quasi-convex if for all $x$ and $y$ in $A$, and all $\lambda \in [0, 1]$, 
  
  $$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}.$$  
  
  (Note that $f$ is quasi-convex iff $-f$ is quasi-concave.)

- Show that $v(p, m)$ is quasi-convex.
  
  - Let $x^\lambda$ maximize $u(x)$ subject to 
    
    $$(\lambda p + (1 - \lambda)p')x \leq \lambda m + (1 - \lambda)m'.$$
  
  - Rearranging terms, we see that 
    
    $$\lambda(px^\lambda - m) + (1 - \lambda)(p'x^\lambda - m') \leq 0.$$  
  
  - So it must be that either $px^\lambda \leq m$ or $p'x^\lambda \leq m'$ (possibly both).
  
  - Therefore either $v(\lambda p + (1 - \lambda)p', \lambda m + (1 - \lambda)m' \leq v(p, m)$
    
    or $v(\lambda p + (1 - \lambda)p', \lambda m + (1 - \lambda)m' \leq v(p', m')$ (Explain why)

- That's it. (Explain why)
Continuity, Berge’s theorem, a.k.a. Theorem of the Maximum

- Parametric constrained maximization problem: Maximize $F(x, a)$ subject to $x \in A(a)$.
  - Let $A(a)$ be a continuous mapping from parameter vectors $a \in \mathbb{R}^n$ to closed bounded subsets of $\mathbb{R}^n$ and the function $F$ be a continuous function.
  - Define the correspondence $x(a) = \{x \in A(a) | F(x, a) \geq F(x', a) \text{ for all } x' \in A(a)\}$
  - Define the function $v(a) = \max_{x \in A(a)} F(x, a)$.

- Then the function $v(a)$ is continuous and the correspondence $x(a)$ is upper semi-continous.
The special case of Berge’s theorem for demand correspondences

- Maximize \( u(x) \) subject to \( px \leq m \) where \( u \) is a continuous, monotone increasing utility function, where \( p >> 0 \) and \( m > 0 \).

- This corresponds to Berge’s theorem with the parameter vector being \( p, m \), the function \( F(x, a) = u(x) \) and the correspondence \( A(p, m) = \{ x \in \mathbb{R}^n_+ | px \leq m \} \) being the budget correspondence.

- The correspondence \( A(p, m) \) turns out to be continuous at all \( (p, m) >> 0 \), so by Berge’s theorem, the demand correspondence \( x(p, m) \) is upper semi-continuous and the indirect utility \( v(p, m) \) is continuous.
Continuous correspondences

- (We will here concern ourselves with correspondences into sets that are non-empty, closed and bounded in $\mathbb{R}^n$.)
- A correspondence $A(a)$ is upper semi-continuous if it has a closed graph. That is, if the sequence $(a_n, x_n) \to (a, x)$ and if $x_n \in A(a_n)$ for all $n$ then $x \in A(a)$. (See picture on the board.)
- A correspondence is lower semi-continuous if for all $x \in A(a)$ and for any sequence $a_n \to a$, we can find $N$ large enough so that for all $n > N$, there is an $x_n \in A(a_n)$ and $x_n \to x$.
- A correspondence is lower semi-continuous if it is both upper and lower semi-continuous.
The budget correspondence is continuous on the set $\mathbb{R}^{n+1}_{++}$.

- Its upper semi-continuous.
- To see this, note that if $(p_n, m_n) \to (p, x)$ and $(p^n x_n \leq m^n)$ for all $n$, then $px \leq m$.
- Its lower semi-continuous
- A bit of algebra shows this.