# Rooftop Theorem for Concave functions

This theorem asserts that if $f$ is a differentiable concave function of a single variable, then at any point $x$ in the domain of $f$, the tangent line through the point $(x, f(x))$ lies entirely above the graph of $f$. You should draw a picture.

**Theorem 1.** *If $f$ is a continuously differentiable concave function of a single variable, defined on a real interval $I$, then for all $x_1$ and $x_2$ in $I$,*

$$f(x_1) + (x_2 - x_1)f'(x_1) \geq f(x_2).$$

Geometrically, this theorem says that the tangent line to the graph of $f$ passing through any point $(x_1, f(x_1))$ must lie entirely on or above the graph of $f$. You should draw a couple of pictures to convince yourself of this geometry.

*Proof.* Since $f$ is a concave function, it must be that for all $x_1$ and $x_2$ in $I$, and all $t \in [0, 1]$,

$$f((1-t)x_1 + tx_2) \geq (1-t)f(x_1) + tf(x_2). \tag{1}$$

Rearranging terms, we see that Equation 1 is equivalent to

$$f(x_1 + t(x_2 - x_1)) - f(x_1) \geq t(f(x_2) - f(x_1)). \tag{2}$$

Dividing both sides of equation 2 by $t$, we have

$$\frac{f(x_1 + t(x_2 - x_1)) - f(x_1)}{t} \geq f(x_2) - f(x_1) \tag{3}$$

Since the inequality in Equation 3 holds for all $t \in [0, 1]$, it follows that

$$\lim_{t \to 0} \frac{f(x_1 + t(x_2 - x_1)) - f(x_1)}{t} \geq f(x_2) - f(x_1) \tag{4}$$

From the definition of a derivative and the chain rule it follows that

$$\lim_{t \to 0} \frac{f(x_1 + t(x_2 - x_1)) - f(x_1)}{t} = (x_2 - x_1)f'(x_1) \tag{5}$$

It follows that:

$$(x_2 - x_1)f'(x_1) \geq f(x_2) - f(x_1). \tag{6}$$

Rearranging Equation 6, we have the desired result, namely

$$f(x_1) + (x_2 - x_1)f'(x_1) \geq f(x_2). \tag{7}$$

□

Now an easy and important consequence of the Rooftop Theorem is the following.

**Theorem 2.** *If $f$ is a continuously differentiable function of a single variable, defined on a real interval $I$, then $f$ is a concave function if and only if $f''(x) \leq 0$ for all $x \in I$.*

One proof of this theorem is to apply Taylor's theorem and the Rooftop theorem. (Hint: Write the exact form of the second order Taylor's expansion.)

Here is another proof. Suppose that $f$ is a concave function. Choose any two points $x$ and $y$ in $I$ such that $x > y$. The Rooftop Theorem implies that $f(x) - f(y) \leq f'(y)(x - y)$ and also $f(y) - f(x) \leq f'(x)(y - x)$. The second inequality is equivalent to $f(x) - f(y) \geq f'(x)(x - y)$. It follows that $f'(x)(x - y) \leq f(x) - f(y) \leq f'(y(x - y)$ and hence that $f'(x) \leq f'(y)$ whenever $x > y$ . But this means that $f'$ is a non-increasing function and hence $f''(x) \leq 0$ for all $x \in I$.

A similar argument establishes the converse.