Question 1

Harry consumes just one commodity and he will live for T periods. His current preferences over consumption streams are represented by a utility function of the form

$$U(x_1, \ldots, x_T) = \sum_{t=1}^{T} \beta_t u(x_t)$$

where $$x_t$$ is the amount of the commodity that he will consume in year t and where the function $$u(\cdot)$$ is strictly concave and twice continuously differentiable. Harry knows that his income stream will be $$(w_1, \ldots, w_T)$$ where $$w_t$$ is the income that he will receive in period t. Harry is able to borrow or lend at the constant interest r. At time 1, Harry is able to commit himself to any time path of consumption that satisfies his budget constraint. His budget constraint is that the present value of his lifetime consumption must not exceed the present value of his lifetime income stream.

Part 1) Suppose that for some $$\alpha$$ where $$0 < \alpha < 1$$, and for all $$t = 1, \ldots, T$$, $$\beta_t = \alpha^{t-1}$$. At what interest rate will Harry will choose to consume the same amount of goods in every period of his life? Explain why your answer is correct. Does this interest rate depend on the time path of his income stream? At this interest rate, what can you say about the way in which his borrowing and saving behavior depend on the time path of his income stream.

Part 2) Suppose that $$\beta_2 = \beta_1 = 1$$ and that for $$t = 3, \ldots, T$$, $$\beta_t = \alpha^{t-2}$$. Suppose that at time 1, Harry can commit himself to a time path of future consumption. Qualitatively, how does his time path of consumption depend on the interest rate? For example, at what if any interest rates $$r > 0$$ is his consumption first increasing, then constant, at what interest rates is his consumption always increasing, at what interest rates is his consumption first increasing, then decreasing, etc, etc.

Part 3) Suppose that $$T = 3$$ and Harry’s utility function is

$$U(x_1, x_2, x_3) = \sqrt{x_1} + \sqrt{x_2} + \alpha \sqrt{x_3}.$$  

Harry earns income $$W > 0$$ in period 1, while $$w_t = 0$$ for $$t > 1$$. Suppose also that 

$$\frac{1}{1+r} = \alpha.$$  

If Harry can commit himself to a time path of future consumption at time 1, solve for his choice of $$x_1, x_2, and x_3$$ as a function of the parameters $$W$$ and r.

Part 4) Suppose that Harry can save money in period 1 but he must leave the choice of allocation between periods 2 and 3 to his future self. Harry is aware
of this and knows that in Period 2 his utility function for consumptions periods 2 and 3 will be
\[ U(x_2, x_3) = \sqrt{x_2} + \sqrt{x_3}. \]
He also knows that the interest rate will continue to satisfy the equation
\[ \frac{1}{1 + r} = \alpha. \]
If Harry consumes \( x_1 \) in period 1, what consumptions will his period 2 self choose for periods 2 and 3? Write down an expression for Harry’s utility as a function of \( x_1 \), taking into account the fact that he knows that his period 2 self will determine the division of income between his period 2 self and his period 3 self. Find the optimal choice of \( x_1 \) for Harry. Is this the same as the amount of \( x_1 \) that he would choose in Part 3 above?