Potential Maximization and Coalition Government Formation

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1 Introduction

This paper addresses coalition government formation. We formulate a model that addresses why and when coalition governments form that include more than the minimum number of parties required for a majority. Non minimum winning coalition governments are interesting in our setting since they do not maximize the surplus of the governing parties. This feature of the model captures the idea that coalition governments with broader membership must accommodate a wider spectrum of policy beliefs, and that this diminishes each parties benefit to being in power.

We model coalition government formation as a cooperation formation game (See Qin, 1996). This is a strategic game in which each player independently identifies whom she wishes to cooperate with in a given coalitional game, and payoffs follow a solution imposed on the coalitional game. We begin by establishing results that are generally applicable to cooperation formation games of all types. We then look specifically at the particular problem of coalition government formation by examining a particular cooperation formation game, that we call the *government formation game*.

The government formation game has three players, who are taken to represent political parties. Each party in the game has an exogenously determined policy position and a share of votes that it received in an election. To avoid boring cases, it is assumed that no party has a majority of the votes and that any two parties can form a majority. The coalition of parties that forms the government is the one that controls a majority of the votes cast. The members of the government are entitled to share a surplus, that is interpreted as the value of forming the government. The surplus is depreciated by an amount that depends on the policy choice of the government. Assuming distinct policy positions, adding more parties to a coalition government lowers the value of forming the government as a greater deal of compromising is required to accommodate a wider spectrum of policy positions. The technical implication is that the characteristic function of the coalitional game is not superadditive.

The surplus of the coalition government is divided according to the Myerson value (Myerson, 1977). The Myerson value is a generalization of the Shapley value (Shapley, 1953) that reflects the cooperation (or link) structure within each coalition (see also Aumann and Myerson, 1988). If all the players in a coalition are linked, the Myerson value is the same as the Shapley value, otherwise more weight is given to players that hold special positions in the link structure; for example, an apex.

The attachment of more weight to players with a special position in the link structure makes sense if we interpret links as representing favorable relationships between the players. In the model, two players are linked if and only if each includes the other in the list of players with whom she wishes to cooperate. A player with links to both of the other players included both of them in her proposal for forming a government and both of them included her. Hence, it is likely that they will work together toward mutually beneficial outcomes in the political process. No link between two players means one player excluded the other in her proposal. It is easy to imagine that there will be sore feelings between unlinked players and that such players will get less benefit from being in the government. Hence, it is natural that less weight should be given to players with missing links.

With the proposed method of dividing surplus the government formation game is a potential game, as defined by Shapley and Monderer (1996). A game is a potential game if there is a function, that characterizes for every player, the change in payoffs to any deviation in strategy choice given any fixed strategy profile for the other players. A useful fact about potential games is that only a subset of the Nash equilibria of potential games coincide with the set of strategy profiles that globally maximize the potential. This makes potential maximization useful as a refinement tool.[1] In fact, for the three-player cooperation formation games considered here it turns out that while there may be multiple equilibria that maximize the potential of a given game, all such outcomes have the same cooperation structure. Thus we are able to arrive at a unique prediction on the cooperation structure of the games we consider.

Only two cooperation structures are possible. We show that potential maximization requires either a two-player government or a three-player government with full cooperation (i.e., every pair of players is linked in the cooperation structure). We provide necessary and sufficient conditions for either type of government. In the general version of the theory, these conditions are stated as restrictions on the players' Shapley values.

[1] See Monderer and Shapley 1996, Section 5.

When a particular specification of the characteristic function is assumed, the conditions are expressed in terms of the size of the government surplus and the distances between parties on the policy line.

Why focus on potential maximization? This is justified in part by observation that maximizing the potential has been shown to be a successful predictor in experiments (see Monderer and Shapley, 1996, Section 5). In addition, for potential games every Nash equilibria can be considered as the outcome of a learning process. Consider the following adjustment process: Each time the strategy profile is not a Nash equilibrium let one player deviate to a better strategy. Such a process is called an improvement path (see Monderer and Shapley, 1996), and for potential games it converges to a Nash equilibrium in finite steps. The particular Nash equilibrium that globally maximizes the potential can be justified as the outcome of a modified adjustment process.

Previous papers that model coalition formation as a simultaneous move game include d'Aspremont et al. (1983), Hart and Kurz (1983) and Yi and Shin (1995). Ours approach differs from these works in a number of ways. First, none of them are concerned with coalition government formation and hence they do address issues specific to this application (such as the specification of the characteristic function). A second difference is the way strategy profiles determine coalition formation. In d'Aspremont et al. the strategy choice is simply a yes/no decision. All players that say yes form a coalition. Hart and Kurz propose multiple games. In one coalitions form only by unanimous agreement. In another, all those who propose the same coalition unite, and it does not have to be unanimous. Yi and Shin is similar in that all players who announce the same message unite whether it is unanimous or not. Third, these papers do not consider potential maximization as a refinement tool. Rather, Hart and Kurz select among multiple Nash equilibria by applying the cooperative refinement notion, Strong Nash Equilibrium, while Yi and Shin consider Coalition Proof Nash Equilibrium. With these notions the set of solutions may not be unique and is possibly empty.

An alternative to simultaneous play is to consider sequential games. See, for example, Moldovanu (1992), Chatterjee et al. (1993), Perry and Reny (1994) and Bloch (1996) in the economics literature, and Austin-Smith and Banks (1988) in the political science literature. This approach has the advantage of allowing the players to be farsighted, but the solution is typically sensitive to the ordering of moves, which in most contexts is arbitrary. There is also a large literature on cooperative approaches to solving three-player games of the sort considered here. This literature focuses on *solutions* that identify sets of feasible payoffs that are stable in one sense or another in a bargaining environment. See Kahan and Rapoport (1984) for a survey of this literature and for a discussion on how well various solutions conform to experimental data.

2 Model

Players are denoted by $i \in \{1,2,3\} = N$. The strategy set of player i is $\Pi_i = \{S \subseteq N \mid i \in S\}$. A strategy $\pi_i \in \Pi_i$ is a set of players with whom player i wishes to form links. Let $\Pi = \times_{i \in N} \Pi_i$. Given $\pi \in \Pi$, a link between players i and j forms if $i \in \pi_j$ and $j \in \pi_i$. The undirected bilateral link between players i and j is denoted $i : j$. The set of all (undirected) bilateral links between players is $L = \{i : j \mid i, j \in N\}$. A cooperation structure is a list of undirected bilateral links in L. Given a strategy profile π, the resulting cooperation structure is given by the function $g : \Pi \to L$ where $g(\pi) = \{i : j \mid i \in \pi_j \text{ and } j \in \pi_i\}$.

Given a set of strategies and hence a cooperation structure, the payoffs are determined as follows. First, it is assumed that any coalition has a value that is independent of the cooperation structure. These values are expressed by a characteristic function $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$. Second, the division of $v(S)$ among the members of S is assumed to satisfy two requirements. One is that it be feasible. The other is that the division rule allows the game to have a potential. It turns out that the only division rule that meets these requirements is the Myerson value (See Qin, 1996).

The Myerson value for the game (N, v) with cooperation structure g is denoted $\psi(v, g) = (\psi_i(v, g))_{i \in N}$ It can be constructed using the Shapley value. The Shapley value of a game (N, v) is defined as follows:

$$\phi_i(v) = \sum_{S : S \ni i} \frac{(|S| - 1)!(n - |S|)!}{n!} [v(S) - v(S \backslash i)]$$

for $i \in N$. Let $\phi(v) = (\phi_i(v))_{i \in N}$. Given $g(\pi) \subseteq L$ and $S \subseteq N$, S/g denotes the partition of S into subsets of players that are connected by $g(\pi)$ in S. Given a characteristic function v and $g(\pi) \subseteq L$, $v^{g(\pi)}$ denotes the characteristic function determined according to

$$v^{g(\pi)}(S) = \sum_{R \in S/g} v(R), \ \ S \subseteq N.$$

Myerson (1977) shows that $\psi(v, g(\pi)) = \phi(v^{g(\pi)})$. Let $U_i(\pi) = \psi_i(v, g(\pi))$ denote the payoff of player i under the strategy profile π. Let $\Gamma = \{\Pi_i, U_i\}_{i \in N}$ denote the cooperation formation game.

Remark 1 $U_i(\pi) = U_i(\pi')$ if and only if $g(\pi) = g(\pi')$.

Definition 1 *(Monderer and Shapley 1996, p. 128) A path in* Π *is a sequence* $\gamma = (\pi^0, \pi^1, ...)$ *of strategy profiles such that for every* $\ell \geq 1$ *there exists a unique player denoted* i_ℓ *such that* $\pi^\ell = (\pi_\ell, \pi^{\ell-1}_{-i_\ell})$ *for some* $\pi_\ell \in \Pi_\ell$ *with* $\pi_\ell \neq \pi^{\ell-1}_{i_\ell}$ *(Player* i_ℓ *is the only deviator from* $\pi^{\ell-1}$ *in* π^ℓ*.)*

Given a path $\gamma = (\pi^0, \pi^1, ...)$, let $I(\gamma) = \sum_\ell (U_{i_\ell}(\pi^\ell) - U_{i_\ell}(\pi^{\ell-1})$. Assume $n = 3$. Fix $\pi^0 = (\pi_1^0, \pi_2^0, \pi_3^0)$ with $\pi_i^0 = \{i\}$. For any $\pi \in \Pi$, let $\gamma(\pi) = (\pi^0, \pi^1, ..., \pi^m)$ be a path such that $\pi^m = \pi$. That is, $\gamma(\pi)$ is a path that connects π^0 with π.

Definition 2 *A potential for Γ is a function $P : \Pi \to \mathbb{R}$ such that for any $i \in N$, $\pi \in \Pi$, and $\hat{\pi}^i \in \Pi^i$,*

$$U_i(\hat{\pi}_i, \pi_{-i}) - U_i(\pi) = P(\hat{\pi}_i, \pi_{-i}) - P(\pi).$$

A game is a *potential game* if it has a potential.

Remark 2 By Theorem 2.8 of Monderer and Shapley (1996), $I(\gamma(\pi)) = I(\gamma'(\pi))$ if Γ is a potential game and if $\gamma(\pi)$ and $\gamma'(\pi)$ are paths connecting π^0 with π.

3 Result on Cooperation Formation

In this section we state and prove our main result for cooperation formation games. The result is that only one bilateral link or all three of them can survive the global potential maximization. This is partly due to the fact that $v(S)$ is nonnegative.

Theorem 1 *Let $\pi^* \in \Pi$ be a strategy profile that globally maximizes the potential. If*

$$v(\{i, j\}) \le v(N) + \frac{1}{2}[v(\{i, k\}) + v(\{j, k\})]$$

for $i, j, k \in N$, $i \neq j$, $j \neq k$, $i \neq k$, then $g(\pi^*) = \{1 : 2,\ 1 : 3,\ 2 : 3\}$. Otherwise $g(\pi^*) = \{i : j\}$, where $i, j \in N$ are such that $v(\{i, j\}) = \max\{v(S) : |S| = 2\}$.

Proof. For $\pi \in \Pi$, let $P(\pi) = I(\gamma(\pi))$. Then $P(\pi)$ is a potential for the game (see (2.1) of Monderer and Shapley (1996)). Fix $\pi \in \Pi$.

Case 1: $g(\pi) = \emptyset$.

In this case, $g(\pi^\ell) = g(\pi^{\ell-1})$ for π^ℓ, $\pi^{\ell-1} \in \gamma(\pi)$.

By remark 1,

$$P(\pi) = 0. \tag{1}$$

Case 2: $g(\pi) = \{i : j\}$ for some $i, j \in N$ with $i \neq j$.

In this case, by remarks 1 and 2, we may assume that $\pi_i = \{i : j\}$, $\pi_j = \{i : j\}$, $\pi^1 = (\pi_i, \pi_{-i}^0)$, and $\pi^2 = (\pi_j, \pi_{-j}^1)$. Since $g(\pi^1) = g(\pi^0) = \emptyset$ and $g(\pi^2) = \{i : j\}$, we have $P(\pi) = U_j(\pi^2) - U_j(\pi^1)$. Because $U_j(\pi^1) = 0$ and $U_j(\pi^2) = \frac{1}{2}v(\{i, j\})$,

$$P(\pi) = \frac{1}{2}v(\{i, j\}). \tag{2}$$

Case 3: $g(\pi) = \{i : j,\ i : k\}$ for $i,\ j, k \in N$ with $i \neq j,\ j \neq k,\ i \neq k$.

In this case, by remarks 1 and 2, we may assume $\pi_i = N$, $\pi_j = \{i : j\}$, $\pi_k = \{i : k\}$, and $\gamma(\pi) = (\pi^0, \pi^1, \pi^2, \pi^3)$, where $\pi^1 = (\pi_j, \pi^0_{-j})$, and $\pi^2 = (\pi_k, \pi^1_{-k})$ and $\pi^3 = (\pi_i, \pi^2_{-i})$. Since $g(\pi^1) = g(\pi^2) = \emptyset$ and $g(\pi^3) = \{i : j,\ i : k\}$, we have $U_j(\pi^0) = U_j(\pi^1) = 0$, $U_k(\pi^0) = U_k(\pi^1) = 0$, $U_i(\pi^2) = 0$, and $U_i(\pi^3) = \frac{1}{6}v(\{i,j\}) + \frac{1}{6}v(\{i,k\}) + \frac{1}{3}v(N)$. Thus,

$$P(\pi) = \frac{1}{6}v(\{i,j\}) + \frac{1}{6}v(\{i,k\}) + \frac{1}{3}v(N). \tag{3}$$

Case 4: $g(\pi) = \{1 : 2,\ 1 : 3,\ 2 : 3\}$

In this case it must be true that $\pi_1 = \pi_2 = \pi_3 = N$. By remark 2, we may assume that $\gamma(\pi) = (\pi^0, \pi^1, \pi^2, \pi^3)$ where $\pi^1 = (\pi_i, \pi^0_{-i})$, and $\pi^2 = (\pi_j, \pi^1_{-j})$ and $\pi^3 = (\pi_k, \pi^2_{-k})$. Then, $g(\pi^1) = \emptyset$, $g(\pi^2) = \{i : j\}$, and $g(\pi^3) = \{i : j,\ i : k,\ j : k\}$. This implies that $U_i(\pi^0) = U_i(\pi^1) = 0, U_j(\pi^1) = 0$, $U_j(\pi^2) = \frac{1}{2}v(\{i,j\})$, $U_k(\pi^2) = 0$, and $U_k(\pi^3) = \frac{1}{6}v(\{i,k\}) + \frac{1}{6}v(\{j,k\}) + \frac{1}{3}v(N) - \frac{1}{3}v(\{i,j\})$. We therefore have

$$P(\pi) = \frac{1}{6}[v(\{i,j\}) + v(\{i,k\}) + v(\{j,k\})] + \frac{1}{3}v(N) \tag{4}$$

Now let π^* denote a strategy profile that globally maximizes the potential. Then by (1), (3), and (4), $g(\pi^*) \neq \emptyset$ and $g(\pi^*) \neq \{i : j,\ i : k\}$ for any $i \neq j,\ j \neq k,\ i \neq k$. By (2) and (4), $g(\pi^*) \neq \{i : j\}$ if and only if $v(\{i,j\}) \geq v(\{i,k\})$, $v(\{i,j\}) \geq v(\{j,k\})$, and $v(\{i,j\}) \geq v(N) + \frac{1}{2}[v(\{i,k\}) + v(\{j,k\})]$; $g(\pi^*) = \{1 : 2,\ 1 : 3,\ 2 : 3\}$ if and only if $v(\{i,j\}) \leq v(N) + \frac{1}{2}[v(\{i,k\}) + v(\{j,k\})]$ for $i \neq j,\ j \neq k,\ i \neq k$. ■

It is possible to restate the theorem in terms of the Shapley value. This is useful for interpreting the result, and for working out examples below.

Corollary 1 *Let π^* be the strategy profile that maximizes the potential of the government formation game. If $\phi_1, \phi_2, \phi_3 \geq 0$ then, $g(\pi^*) = \{1 : 2,\ 1 : 3,\ 2 : 3\}$. Otherwise, $g(\pi^*) = \{i : j\}$, where $i, j \in N$ are such that $v(\{i,j\}) = \max\{v(S) : |S| = 2\}$.*

Proof. The restatement of Theorem 1 into Corollary 1 is immediate since the Shapley value for player k in the full cooperation game is $\phi_k = \frac{1}{6}v(\{i,k\}) + \frac{1}{6}v(\{j,k\}) + \frac{1}{3}v(N) - \frac{1}{3}v(\{i,j\})$. ■

Remark. The following fact is useful for checking the conditions of the Theorem/Corollary. Suppose $v(\{i,j\}) = \max\{v(S) : |S| = 2\}$, where $i, j \in N$ and let $k = N/\{i,j\}$. Then $\phi_k \geq 0$ implies $\phi_i \geq 0$ and $\phi_j \geq 0$.

3.1 The Government Formation Game

We consider a parliamentary system in which there are three parties. Each party has an exogenously given policy position represented by a point on the real line. The policy position of each party $i \in N$ is denoted $p_i \in \mathbb{R}$. Each party controls a number of seats that it won in an election. Seat shares are assumed to be such that any two parties (or all three parties) can combine to control a majority of the seats, but no single party has a majority. There is a surplus of G that is received by the members of whichever coalition government forms. However, each party in the coalition government suffers a loss equal the square of the distance between their policy position and that chosen by the coalition government. The government formation game is an example of a cooperation formation game where the characteristic function v is defined by

$$v(S) = \begin{cases} \max_{y \in P} G - \sum_{i \in S} (y - p_i)^2 & \text{if} \quad |S| \geq 2 \\ \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

One thing is obvious. Let $T(p_1, p_2, p_3) = \{(p_1', p_2', p_3') \in \mathbb{R}^3 : (p_1', p_2', p_3') = (p_1, p_2, p_3) + \theta, \theta \in \mathbb{R}\}$. The characteristic function v is invariant to changes in party positions within the set $T(p_1, p_2, p_3)$. In words, the relative policy positions of the parties matter, the absolute policy positions do not.

Notice that because of the specification of v, we have the following implications of Theorem/Corollary 1.

Proposition 1 *A two party government will include the two parties that are closest together on the policy line.*

Proof. Suppose $\phi_k < 0$ for some $k \in N$ so that some two-party government forms. Assume that parties 1 and 2 are closest together on the policy line so that $|p_1 - p_2| \leq |p_r - p_s|$ for all $r, s \in N$. For any two party coalition $\{r, s\}$, $r, s \in N$, the solution to the maximization problem in (5) is $y^* = (p_r + p_s)/2$. Consequently, $v(\{r, s\}) = G - \frac{1}{2}(p_r - p_s)^2$ for any.$r, s \in N$. It is immediate that $v(\{1, 2\}) = \max\{v(S) : |S| = 2\}$, and hence by the theorem a two-party government including parties 1 and 2 forms. ■

Proposition 2 *Assume that parties* 1 *and* 2 *are closest together on the policy line. Let* π^* *maximize the potential function of the government formation game. Then a three party government will form if and only if* $\frac{7}{12}(p_2 - p_3)^2 + \frac{7}{12}(p_1 - p_3)^2 - \frac{1}{6}(p_1 - p_2)^2 \leq G$.

Proof. The solution to the maximization problem in (5) is

$$y^* = \begin{cases} \frac{p_r + p_s}{2} & \text{if } S = \{r, s\} \text{ for any } r, s \in N; \\ \\ \frac{p_1 + p_2 + p_3}{3} & \text{if } S = N. \end{cases}$$

Using y^* in (5),we have $v(\{r,s\}) = G - \frac{1}{2}(p_r - p_s)^2$ for any $r, s \in N$. Since, $|p_1 - p_2| \leq |p_r - p_s|$ for all $r, s \in N$, it is immediate that $v(\{1,2\}) = \max\{v(S) : |S| = 2\}$. Moreover,

$$\begin{aligned}\phi_3 &= \frac{1}{6}v(\{1,3\}) + \frac{1}{6}v(\{2,3\}) + \frac{1}{3}v(N) - \frac{1}{3}v(\{1,2\}) \\ &= \frac{1}{6}(G - \frac{1}{2}(p_1 - p_3)^2) + \frac{1}{6}(G - \frac{1}{2}(p_2 - p_3)^2) \\ +\frac{1}{3}(G - (\frac{p_2 + p_3 - 2p_1}{3})^2 &- (\frac{p_1 + p_3 - 2p_2}{3})^2 - (\frac{p_1 + p_2 - 2p_3}{3})^2) \\ &- \frac{1}{3}(G - \frac{1}{2}(p_1 - p_2)^2).\end{aligned}$$

Factoring yields

$$\phi_k = \frac{1}{3}[G - \frac{7}{12}(p_2 - p_3)^2 - \frac{7}{12}(p_1 - p_3)^2 + \frac{1}{6}(p_1 - p_2)^2].$$

The result follows by Corollary 1. ■

Replace the three policy position variables with three distance variables $x = |p_2 - p_3|$, $y = |p_1 - p_3|$, and $z = |p_1 - p_2|$. Then, the quadratic surface described in the above proposition is a hyperboloid of one sheet. For each surface, the trace in a plane $z =$ *constant* is a circle, and the traces in the planes $x =$ *constant* and $y =$ *constant* are hyperboloids of revolution. Since the policy space is one dimensional, one distance is the sum of the other two. In particular, suppose

$$p_1 \leq p_2 \leq p_3. \tag{6}$$

If

$$p_2 - p_1 \leq p_3 - p_2 \tag{7}$$

then the three-variable[2] second-degree equation in the above proposition can be written as a two-variable, second-degree equation of the form

$$\frac{7}{6}(p_3 - p_2)^2 + \frac{7}{6}(p_3 - p_2)(p_2 - p_1) + \frac{5}{12}(p_2 - p_1)^2 \leq G. \tag{8}$$

Otherwise, if

$$p_2 - p_1 \geq p_3 - p_2 \tag{9}$$

[2] The variables we refer to are x, y and z.

then the three-variable second-degree equation in the proposition can be written as a two-variable, second-degree equation of the form

$$\frac{7}{6}(p_2 - p_1)^2 + \frac{7}{6}(p_3 - p_2)(p_2 - p_1) + \frac{5}{12}(p_3 - p_2)^2 \leq G. \tag{10}$$

In either case, the quadratic surface is a rotated ellipse. The intersection of the two ellipses in the set $\{(x, z) : x = p_3 - p_2 \geq 0,\ z = p_2 - p_1 \geq 0\}$ is shown in Figure 1, below.

Figure 1: $G = 1$

The set of points satisfying (6), (7), and (8) is shown as quadrant A. The set of points satisfying (6), (9), and (10) is shown as quadrant B. In quadrants A and B, a three-party government forms. Parties 1 and 2 form a two-party government in quadrant C, while 2 and 3 form a two-party government in quadrant D. The predictions of the theory are sensible. At points in quadrants A and B the three parties are close together and hence it is reasonable to expect that a three party government will form. At points in quadrant C or D the parties are more disperse, in the sense that at least one of the parties is far from the other two.

3.2 Discussion

It is immediate from condition (ii) of the theorem that the grand coalition (with a complete graph) forms the government if the game (N, v) is superadditive. This is established previously in Qin (1996).[3] Moreover, it is interesting to note that for non superadditive games the theorem does not predict two party-governments in all cases. Hence, coalition governments may form that do not maximize the surplus that is split by the governing members. In the case of non superadditive games, the formation of the two-party government that satisfies $v(\{i, j\}) = \max\{v(S) : |S| = 2\}$ is the efficient outcome, in the sense that it maximizes total surplus. The formation of any other two-party government or a three party government is an inefficient outcome. Examples that provide insight into why we might expect occurrences of inefficient coalition governments are provided in the next section.

Some may find it surprising that the magnitude of G matters in determining whether the government will be two-party or three-party. This occurs because the Myerson value averages marginal contributions that occur when a party enters a coalition that is already winning with cases where the coalition is not already winning. Whenever a party joins a coalition that is already winning it reduces the surplus available its members. Whether or not such instances are overridden by cases where the party is instrumental in causing a coalition to be winning, depends on the magnitude of G. The practical implication is that we expect larger governments to be more prevalent in instances where the gains to being in power are large.

4 Examples

4.1 Three parties with the same policy positions

First, we present an example where all three parties have the same policy position and $G = 1$. Then the characteristic function reduces to that of a simple majority voting game with $v(i) = 0$, $i \in N$, and $v(S) = 1$, $|S| \geq 2$. This game is superadditive.

The Myerson values for different cooperation structures are as follows: If $g = \{i : j\}$ then $\psi(i) = \psi(j) = 1/2$, $\psi(k) = 0$. If $g = \{i : k\}$ then $\psi(i) = \psi(k) = 1/2$, $\psi(j) = 0$. If $g = \{j : k\}$ then $\psi(i) = 0$, $\psi(j) = \psi(k) = 1/2$. If $g = \{i : j, i : k\}$ then $\psi(i) = 2/3$, $\psi(j) = \psi(k) = 1/6$. If $g = \{i : j, j : k\}$ then $\psi(i) = \psi(k) = 1/6$, $\psi(j) = 2/3$. If $g = \{i : k, j : k\}$ then $\psi(i) = \psi(j) = 1/6$, $\psi(k) = 2/3$. If $g = \{1 : 2, 1 : 3, 2 : 3\}$ then $\psi(1) = \psi(2) = \psi(3) = 1/3$. Based on these values, the payoff matrices for each fixed strategy of player k are given below.

[3] See Dutta, B., A. van den Nouweland, and S. Tijs (1998), Slikker et. al. (1998), and Slikker (2000) for additional results pertaining to cooperation formation in superadditive games.

$\boldsymbol{\pi}_3 = \{\mathbf{3}\}$

$\pi_1 \backslash \pi_2$	$\{2\}$	$\{1,2\}$	$\{2,3\}$	N
$\{1\}$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
$\{1,2\}$	$(0,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(0,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$
$\{1,3\}$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
N	$(0,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(0,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$

$\boldsymbol{\pi}_3 = \{\mathbf{1},\mathbf{3}\}$

$\pi_1 \backslash \pi_2$	$\{2\}$	$\{1,2\}$	$\{2,3\}$	N
$\{1\}$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
$\{1,2\}$	$(0,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(0,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$
$\{1,3\}$	$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{1}{2},0,\frac{1}{2})$
N	$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{2}{3},\frac{1}{6},\frac{1}{6})$	$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{2}{3},\frac{1}{6},\frac{1}{6})$

$\boldsymbol{\pi}_3 = \{\mathbf{2},\mathbf{3}\}$

$\pi_1 \backslash \pi_2$	2	$\{1,2\}$	$\{2,3\}$	N
$\{1\}$	$(0,0,0)$	$(0,0,0)$	$(0,\frac{1}{2},\frac{1}{2})$	$(0,\frac{1}{2},\frac{1}{2})$
$\{1,2\}$	$(0,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(0,\frac{1}{2},\frac{1}{2})$	$(\frac{1}{6},\frac{2}{3},\frac{1}{6})$
$\{1,3\}$	$(0,0,0)$	$(0,0,0)$	$(0,\frac{1}{2},\frac{1}{2})$	$(0,\frac{1}{2},\frac{1}{2})$
N	$(0,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(0,\frac{1}{2},\frac{1}{2})$	$(\frac{1}{6},\frac{2}{3},\frac{1}{6})$

$\boldsymbol{\pi}_3$= N

$\pi_1 \backslash \pi_2$	$\{2\}$	$\{1,2\}$	$\{2,3\}$	N
$\{1\}$	$(0,0,0)$	$(0,0,0)$	$(0,\frac{1}{2},\frac{1}{2})$	$(0,\frac{1}{2},\frac{1}{2})$
$\{1,2\}$	$(0,0,0)$	$(\frac{1}{2},\frac{1}{2},0)$	$(0,\frac{1}{2},\frac{1}{2})$	$(\frac{1}{6},\frac{2}{3},\frac{1}{6})$
$\{1,3\}$	$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{1}{6},\frac{1}{6},\frac{2}{3})$	$(\frac{1}{6},\frac{1}{6},\frac{2}{3})$
N	$(\frac{1}{2},0,\frac{1}{2})$	$(\frac{2}{3},\frac{1}{6},\frac{1}{6})$	$(\frac{1}{6},\frac{1}{6},\frac{2}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$

The potential function is

$$P = \begin{array}{lcccc}
\pi_1, \pi_3 \backslash \pi_2 & \{2\} & \{1,2\} & \{2,3\} & N \\
\{1\},\{3\} & 1 & 1 & 1 & 1 \\
\{1\},\{1,3\} & 1 & 1 & 1 & 1 \\
\{1\},\{2,3\} & 1 & 1 & 1\frac{1}{2} & 1\frac{1}{2} \\
\{1\},N & 1 & 1 & 1\frac{1}{2} & 1\frac{1}{2} \\
\{1,2\},\{3\} & 1 & 1\frac{1}{2} & 1 & 1\frac{1}{2} \\
\{1,2\},\{1,3\} & 1 & 1\frac{1}{2} & 1 & 1\frac{1}{2} \\
\{1,2\},\{2,3\} & 1 & 1\frac{1}{2} & 1\frac{1}{2} & 1\frac{2}{3} \\
\{1,2\},N & 1 & 1\frac{1}{2} & 1\frac{1}{2} & 1\frac{2}{3} \\
\{1,3\},\{3\} & 1 & 1 & 1 & 1 \\
\{1,3\},\{1,3\} & 1\frac{1}{2} & 1\frac{1}{2} & 1\frac{1}{2} & 1\frac{1}{2} \\
\{1,3\},\{2,3\} & 1 & 1 & 1\frac{1}{2} & 1\frac{1}{2} \\
\{1,3\},N & 1\frac{1}{2} & 1\frac{1}{2} & 1\frac{2}{3} & 1\frac{2}{3} \\
N,\{3\} & 1 & 1\frac{1}{2} & 1 & 1\frac{1}{2} \\
N,\{1,3\} & 1\frac{1}{2} & 1\frac{2}{3} & 1\frac{1}{2} & 1\frac{2}{3} \\
N,\{2,3\} & 1 & 1\frac{1}{2} & 1\frac{1}{2} & 1\frac{2}{3} \\
N,N & 1\frac{1}{2} & 1\frac{2}{3} & 1\frac{2}{3} & 1\frac{5}{6}
\end{array}$$

The set of strategies that maximizes the potential function is (N,N,N), and the payoffs are $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$.

4.2 Two the same and one different

Suppose two agents 1 and 2 have identical policy positions and one agent 3 has a different one. Let d denote the **squared difference** between 1's policy position and 3's (or equivalently, 2's policy position and 3's). The characteristic function in equation (5) has $v(1) = v(2) = v(3) = 0$, $v(\{1,2\}) = G$, $v(\{1,3\}) = v(\{2,3\}) = G - \frac{d}{2}$, and $v(N) = G - \frac{2d}{3}$. Note that the game is not superadditive.

The Myerson values for different cooperation structures are as follows: If $g = \{1:2\}$ then $\psi(1) = \psi(2) = G/2$, $\psi(3) = 0$. If $g = \{1:3\}$ then $\psi(1) = \psi(3) = G/2 - d/4$, $\psi(2) = 0$. If $g = \{2:3\}$ then $\psi(1) = 0$, $\psi(2) = \psi(3) = G/2 - d/4$. If $g = \{1:2,\ 1:3\}$ then $\psi(1) = 2G/3 - 33d/108$, $\psi(2) = G/6 - d/18$, $\psi(3) = G/6 - 33d/108$. If $g = \{1:2,\ 2:3\}$ then $\psi(1) = G/6 - d/18$, $\psi(2) = 2G/3 - 33d/108$, $\psi(3) = G/6 - 33d/108$. If $g = \{1:3,\ 2:3\}$ then $\psi(1) = \psi(2) = G/6 - 5d/36$, $\psi(3) = 2G/3 - 7d/18$. If $g = \{1:2,\ 1:3,\ 2:3\}$ then $\psi(1) = \psi(2) = G/3 - 5d/36$, $\psi(3) = G/3 - 7d/18$. The payoff matrices for each fixed strategy of player 3 are given below.

$\boldsymbol{\pi}_3 = \{\mathbf{3}\}$

$\pi_1 \backslash \pi_2$	2	$\{1,2\}$	$\{2,3\}$	N
1	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
$\{1,2\}$	$(0,0,0)$	$(\frac{G}{2},\frac{G}{2},0)$	$(0,0,0)$	$(\frac{G}{2},\frac{G}{2},0)$
$\{1,3\}$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
N	$(0,0,0)$	$(\frac{G}{2},\frac{G}{2},0)$	$(0,0,0)$	$(\frac{G}{2},\frac{G}{2},0)$

$\boldsymbol{\pi}_3 = \{\mathbf{1,3}\}$

$\pi_1 \backslash \pi_2$	2	$\{1,2\}$	$\{2,3\}$	N
1	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
$\{1,2\}$	$(0,0,0)$	$(\frac{G}{2},\frac{G}{2},0)$	$(0,0,0)$	$(\frac{G}{2},\frac{G}{2},0)$
$\{1,3\}$	$(\frac{2G-d}{4},0,\frac{2G-d}{4})$	$(\frac{2G-d}{4},0,\frac{2G-d}{4})$	$(\frac{2G-d}{4},0,\frac{2G-d}{4})$	$(\frac{2G-d}{4},0,\frac{2G-d}{4})$
N	$(\frac{2G-d}{4},0,\frac{2G-d}{4})$	$(\frac{24G-11d}{36},\frac{3G-d}{18},\frac{6G-11d}{36})$	$(\frac{2G-d}{4},0,\frac{2G-d}{4})$	$(\frac{24G-11d}{36},\frac{3G-d}{18},\frac{6G-11d}{36})$

$\pi_3 = \{2,3\}$

$\pi_1 \backslash \pi_2$	2	$\{1,2\}$	$\{2,3\}$	N
1	$(0,0,0)$	$(0,0,0)$	$(0, \frac{2G-d}{4}, \frac{2G-d}{4})$	$(0, \frac{2G-d}{4}, \frac{2G-d}{4})$
$\{1,2\}$	$(0,0,0)$	$(\frac{G}{2}, \frac{G}{2}, 0)$	$(0, \frac{2G-d}{4}, \frac{2G-d}{4})$	$(\frac{3G-d}{18}, \frac{24G-11d}{36}, \frac{6G-11d}{36})$
$\{1,3\}$	$(0,0,0)$	$(0,0,0)$	$(0, \frac{2G-d}{4}, \frac{2G-d}{4})$	$(0, \frac{2G-d}{4}, \frac{2G-d}{4})$
N	$(0,0,0)$	$(\frac{G}{2}, \frac{G}{2}, 0)$	$(0, \frac{2G-d}{4}, \frac{2G-d}{4})$	$(\frac{3G-d}{18}, \frac{24G-11d}{36}, \frac{6G-11d}{36})$

$\pi_3 = \mathbf{N}$

$\pi_1 \backslash \pi_2$	2	$\{1,2\}$	$\{2,3\}$	N
1	$(0,0,0)$	$(0,0,0)$	$(0, \frac{2G-d}{4}, \frac{2G-d}{4})$	$(0, \frac{2G-d}{4}, \frac{2G-d}{4})$
$\{1,2\}$	$(0,0,0)$	$(\frac{G}{2}, \frac{G}{2}, 0)$	$(0, \frac{2G-d}{4}, \frac{2G-d}{4})$	$(\frac{3G-d}{18}, \frac{24G-11d}{36}, \frac{6G-11d}{36})$
$\{1,3\}$	$(\frac{2G-d}{4}, 0, \frac{2G-d}{4})$	$(\frac{2G-d}{4}, 0, \frac{2G-d}{4})$	$(\frac{6G-5d}{36}, \frac{6G-5d}{36}, \frac{12G-7d}{18})$	$(\frac{6G-5d}{36}, \frac{6G-5d}{36}, \frac{12G-7d}{18})$
N	$(\frac{2G-d}{4}, 0, \frac{2G-d}{4})$	$(\frac{24G-11d}{36}, \frac{3G-d}{18}, \frac{6G-11d}{36})$	$(\frac{6G-5d}{36}, \frac{6G-5d}{36}, \frac{12G-7d}{18})$	$(\frac{12G-5d}{36}, \frac{12G-5d}{36}, \frac{6G-7d}{18})$

Proposition 3 *The potential is maximized at (N, N, N) for $G \geq \frac{7d}{6}$. Moreover, playing N is a dominant strategy for all three players if and only if $G \geq \frac{11d}{6}$. If $G < \frac{11d}{6}$ playing N is not a dominant strategy for any of the players.*

Proof. Here $p_1 = p_2 = p$ and $p_3 = q$. So the condition in Proposition 2 becomes $\frac{7}{12}(p-q)^2 + \frac{7}{12}(p-q)^2 = \frac{7d}{6} \leq G$. Thus, the potential is maximized at (N, N, N) for $G \geq \frac{7d}{6}$. Establishing the validity of the statements about dominant strategies is simply a matter of checking all of the required restrictions on payoffs in the above payoff matrix. This is left to the interested reader. ■

Consider $d = 1$ and $G = \frac{3}{2}$. For these parameter values $\frac{7d}{6} < G < \frac{11d}{6}$. There are no dominant strategies, but there are eight Nash equilibria corresponding to six different graphs. These are presented below.

	π	$g(\pi)$
I	$(\{1\},\{2\},\{3\})$	$\emptyset$
IIa	$\{1,2\},\{1,2\},\{3\}$	$\{1:2\}$
IIb	$(N,\{1,2\},\{3\})$	$\{1:2\}$
IIc	$(\{1,2\},N,\{3\})$	$\{1:2\}$
III	$(\{1,3\},\{2\},\{1,3\})$	$\{1:3\}$
IV	$(\{1\},\{2,3\},\{2,3\})$	$\{2:3\}$
V	$(\{1,3\},\{2,3\},N)$	$\{1:3,\ 2:3\}$
VI	(N,N,N)	$\{1:2,\ 1:3,\ 2:3\}$

The theoretical prediction is Nash equilibria VI corresponding to the complete graph.[4] It would be interesting to see if this is outcome emerges in experiments.

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