Solving Consumer Choice Problems

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The general, two-variable consumer choice problem involves maximizing a consumer’s utility, subject to a budget constraint:

$$\max U(X,Y) \quad \text{s.t.} \quad p_X X + p_Y Y = I$$

The purpose of the problem is to find one or both of the demand functions $X(p_X, p_Y, I)$ and $Y(p_X, p_Y, I)$ or, when the prices and income are given particular values, a particular bundle $(X^*, Y^*)$. Solving the problem boils down to solving a system of two equations:

1. The budget equation.
2. The optimality condition.

The budget line is typically simple to derive within a given problem; it is just a matter of setting spending to income. The optimality condition, however, is not so straightforward, as it and the method for finding it, depends upon the utility function. What follows is an outline describing how to derive the optimality rule for some typical utility functions studied in intermediate microeconomics.

Case I, Perfect Substitutes: $U(X,Y) = \alpha X + \beta Y$

This utility function is differentiable, so one might be tempted to solve for the optimal bundle by setting the absolute value of $MRS_{X,Y}$ equal to the ratio of the prices (the tangency condition between indifference curves and the budget equation). Doing so will either yield an equation that does not provide information about $X$ or $Y$, and is often nonsensical (e.g. suppose $\alpha = \beta = p_X = 1$ and $p_Y = 2$, then setting $MRS_{X,Y} = \alpha/\beta = 1$ equal to $p_X/p_Y = 1/2$ yields a statement that is not true). When setting $|MRS_{X,Y}|$ equal to the ratio of the prices yields a true statement, then the budget line and the consumer’s indifference curves overlap and any bundle on the consumer’s budget line is optimal. When there is an inequality between $|MRS_{X,Y}|$ and the ratio of the prices, one can utilize this information to deduce the optimal consumption bundle. Suppose, for example, that $|MRS_{X,Y}| < p_X/p_Y$, then this implies that $(MU_X/p_X) < (MU_Y/p_Y)$, which shows that the marginal utility per unit of income spent on good $X$ is less than that spent on good $Y$; the consumer gets more “bang for the buck” by spending money on good $Y$. Thus, spending all income on $Y$ and none on $X$ is optimal; the optimality rule is:

$$X^* = 0, \quad Y^* = I/p_Y.$$

When $MU_X/p_X < MU_Y/p_Y$, it can be shown, using similar reasoning, that:

$$X^* = I/p_X, \quad Y^* = 0.$$
Case II, Perfect Compliments: \(U(X,Y) = \min\{\alpha X, \beta Y\}\)

For perfect compliments, analyzing the relationship between \(|MRS_{X,Y}|\) and the ratio of the prices will be unproductive, as \(|MRS_{X,Y}|\) is equal to infinity, zero, or does not exist. Simple logic can be utilized to deduce the optimality rule. First suppose that a consumer with these preferences consumes a bundle such that \(\alpha X > \beta Y\). Is this consumer utilizing his or her resources wisely? No, spending less on good \(X\) and more on \(Y\) would yield greater satisfaction, and vice versa for the case when \(\alpha X < \beta Y\). Thus, the optimality rule must be that \(\alpha X = \beta Y\). (More generally, when \(U(X,Y) = \min\{f(X, Y), \ g(X, Y)\}\), where \(f\) and \(g\) are both increasing in both variables, the optimality condition is \(f(X, Y) = g(X, Y)\).

Case III, Cobb-Douglas: \(U(X,Y) = AX^\alpha Y^\beta\)

For Cobb-Douglas preferences, indifference curves never intersect the axes, so corner solutions (in which the level of one good or another is zero) are never optimal. Furthermore, \(U\) is differentiable and yields a diminishing \(|MRS_{X,Y}|\). These two characteristics imply that setting the \(|MRS_{X,Y}|\) equal to the ratio of the prices will yield the optimal bundle.

Case IV, Quasilinear: \(U(X,Y) = Y + f(X)\), when \(f' > 0\) and \(f'' < 0\), i.e. \(f\) is concave.

With quasilinear preferences, it is advisable to initially analyze the relationship between \(|MRS_{X,Y}|\) and the ratio of the prices. If setting these two equal to each other yields a bundle that is feasible, then the optimal bundle has been found (assuming no mistakes were made). Often, however, this method yields solutions with some good being less than zero, which is usually not possible (how do you consume negative of some good?). In this case, one should check for corner solutions, by setting one or another good equal to zero and the other equal to income divided by the price. The corner solution that yields the highest level of utility is the optimal bundle. (It is always the case that whichever good was found to be negative by setting \(|MRS_{X,Y}|\) equal to the ratio of the prices should be the good that is not consumed in a corner solution under quasilinear preferences.)

Summary of Optimality Rules:

**Perfect Substitutes:** \(U(X,Y) = \alpha X + \beta Y\)

if \(|MRS_{X,Y}| = \alpha/\beta < p_x/p_y\), then \(X^* = 0, Y^* = I/p_Y\)

\(|MRS_{X,Y}| = \alpha/\beta > p_x/p_y\), then \(X^* = I/p_X, Y^* = 0\)

\(|MRS_{X,Y}| = \alpha/\beta = p_x/p_y\), then any bundle on budget line is optimal.

**Perfect Compliments:** \(U(X,Y) = \min\{\alpha X, \beta Y\}\)

\(\alpha X = \beta Y\)

**Cobb-Douglas:** \(U(X,Y) = AX^\alpha Y^\beta\)
\[ |MRS_{X,Y}| = \alpha Y / \beta X = p_x / p_y \]

Quasilinear: \( U(X,Y) = Y + f(X); f \) is concave.

\[ |MRS_{X,Y}| = \frac{df(X)}{dX} = p_x / p_y, \] unless this condition yields a negative number for one of the goods, in which case the optimal bundle is a corner solution, with the good that the tangency condition requires to be negative equal to zero.